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Theta Modular Groups Determined by Point Sets.

By Arthur B. Coble.*

Introduction.

In a series of articles already published† by the writer, the study of the properties of a set P_n^k of n discrete points in S_k has been initiated. In these articles particular attention has been paid to certain discontinuous groups defined by the P_n^k, to the invariants of these groups, and to applications to the theory of equations. During the course of the work there appeared, in a special case, a connection between the point set and theta modular functions. It is the purpose of this article to show that this connection must exist in the general case.

An especially interesting type of point set is the self-associated P_{2p+2}^p.‡ This set of $2p+2$ points in S_p has the characteristic geometric property that all the quadrics on $2p+1$ points of the set pass through the remaining point. It has the characteristic algebraic property that complementary determinants formed from the matrix of the coordinates of the points differ by a fixed factor of proportionality. If $p+2$ of the points be chosen as a base in S_p, the remaining p points lie on an S_{p-1}, α, in S_p, and quadrics on the base cut α in quadric sections, all of which are apolar to a quadric Q_α in α. The p points on α are any self-polar p-edron of Q_α. After choosing the base in S_p, p absolute constants are required to fix α, and $\frac{1}{2}p(p-1)$ absolute constants are required to fix the self-polar p-edron of Q_α, whence the number of absolute constants in the self-associated P_{2p+2}^p is $\frac{1}{2}p(p+1)$.

Thus the number of absolute constants of this self-associated point set and the number of moduli of the general theta function of p variables are the same. This might be dismissed as a mere coincidence were it not for the fact that the point set and the moduli are known to be related in a number of

* This investigation has been carried on under the auspices of the Carnegie Institution of Washington, D. C.

† "Point Sets and Cremona Groups," Parts I, II, III, *Transactions of the American Mathematical Society*, 1915-16-17. These are referred to hereafter as P. S. I, II, III respectively.

‡ P. S. I (12).

special cases. A general class of such cases is the set of $2p+2$ points on the rational norm curve R^p in S_p. Obviously all the quadrics on $2p+1$ of these points will pass through the remaining one, and the set is self-associated. Let us call it a *hyperelliptic* self-associated set. It is well known that such a set defines the hyperelliptic theta functions with $2p-1$ independent moduli. Nor is the relation here existing peculiar to the hyperelliptic case. For when p is 3, the smallest value of p for which the theta functions are not necessarily hyperelliptic, it is well known that the self-associated set of eight points in S_3 will define a quartic curve of genus 3, and thereby also the moduli of the theta functions of genus 3. In all of these cases the properties of the point set and of the modular functions have been developed sufficiently to make the connection quite clear.*

In P. S. I § 6, projectively equivalent ordered sets P_n^k were mapped upon a point P of a space $\Sigma_{k(n-k-2)}$ and by permutation of the points of P_n^k a group $G_{n!}$ in Σ was derived. A conjugate set of points P in Σ under $G_{n!}$ represents all sets P_n^k projective in some order to each other. In P. S. II § 3, the congruence of sets P_n^k was defined. Two point sets P_n^k and $P_n'^k$ are congruent if the one arises from the other by means of a sequence of operations which are either projectivities or those particular Cremona transformations which occur when the variables are inverted. In applying an operation of the latter type it is understood that the $k+1$ F-points of the Cremona transformation belong to both congruent sets P_n^k and $P_n'^k$, and that the remaining $n-(k+1)$ points of each set form corresponding pairs of the Cremona transformation. It then appeared that all point sets congruent in some order to a given set P_n^k were mapped in Σ by points P which formed a conjugate set under a group $G_{n,k}$ in $\Sigma_{k(n-k-2)}$. This infinite, discontinuous Cremona group $G_{n,k}$ is thereby projectively defined by the point set P_n^k. In the main this article is concerned with the structure of this group.

Though generating elements of $G_{n,k}$ have been given (P. S. II(15)), it is more convenient to handle certain isomorphic groups of linear transformations, particularly the group $g_{n,k}$. The operation which transforms a point set P_n^k into a congruent set $P_n'^k$ will transform an algebraic spread in S_k of order x_0, and multiplicities $x_1, \ldots, x_n$ at the points of P_n^k into an algebraic spread of order x_0' with multiplicities $x_1', \ldots, x_n'$ at the points of $P_n'^k$. Then x' is the linear transform of x under an element of $g_{n,k}$. The group $g_{n,k}$ is simply

* For the case $p=3$, however, explicit formulae for the invariants of the point set in terms of the modular functions are still lacking.

isomorphic with $G_{n,k}$ except in a few particular cases with which we are not concerned here. The elements of $g_{n,k}$ have determinants ± 1 and integer coefficients. Hence the group $g_{n,k}$ admits of a modular theory precisely similar to that of the groups of transformations of the periods of the multiply periodic functions. For example, it is clear that

(1) *The elements of $g_{n,k}$ whose coefficients reduce modulo m to those of the identity, form an infinite invariant subgroup of $g_{n,k}$ whose factor group $g^{(m)}_{n,k}$ is finite.*

We shall determine in this article the order and the structure of all the groups $g^{(2)}_{n,k}$. Since $g_{n,k}$ has a quadratic invariant we should expect these groups to be subgroups of the group of a null system in the finite geometry mod. 2.

In § 1 the generators and invariants of $g_{n,k}$ are given, and these are reduced mod. 2, to generators and invariants respectively of $g^{(2)}_{n,k}$. Certain new invariants are derived and the sets of linear forms conjugate under $g^{(2)}_{n,k}$ are tabulated. These linear forms are permuted under $g^{(2)}_{n,k}$ just as certain geometric objects in the finite geometry of the null system are permuted under certain subgroups of the group of the null system. The identification of the forms and the corresponding geometric objects is made in § 3, there being sixteen cases according as

$$k=4\lambda+\varkappa, \quad n=4\mu+\nu \quad (\varkappa, \nu=0, 1, 2, 3). \tag{2}$$

The finite geometry is exhibited in terms of the basis notation. This notation is recapitulated in § 2, and is there used to determine the structure of the groups which appear in § 3.*

In order to impose conveniently the conditions for a self-associated set, we use the *ultra-elliptic* point set P^k_n, *i. e.*, n-points on the normal elliptic curve E^{k+1} in S_k with elliptic parameters $u_1, \ldots, u_n$. Thus in S_p the $2p+2$-points cut out on E^{p+1} by a quadric evidently constitute an ultra-elliptic self-associated set, and the condition for self-association is merely

$$u_1+u_2+\ldots+u_{2p+2}\equiv 0. \tag{3}$$

For $p=2$ and $p=3$ every self-associated set is ultra-elliptic, and it can be shown that this is true for $p=4$ also. For further values of p the ultra-elliptic self-associated set must be special since it contains only $2p+2$ absolute constants—one for the E^{p+1} and $2p+1$ for the parameters of all but one of the points.

*The statements are drawn from two earlier papers of the writer: "An Application of Finite Geometry to the Characteristic Theory of the Odd and Even Theta Functions," *Trans. Amer. Math. Soc.*, Vol. XIV (1913), and "An Isomorphism between Theta Characteristics and the $(2p+2)$-point," *Annals of Math.*, Vol. XVII, Ser. 2 (1916). These are referred to respectively as F. G. I and F. G. II.

If an ultra-elliptic set P_n^k on E^{k+1} be transformed into a congruent ultra-elliptic set $P_n'^k$ on E'^{k+1}, the two elliptic curves are projective. If E'_{k+1} be projected upon E^{k+1} the set $P_n'^k$ is projected upon a set on E^{k+1} with parameters $u'_1, \ldots, u'_n$. Then the u''s and u's are linearly related under an element of the group $e_{n,k}$ which is simply isomorphic with $g_{n,k}$ (P. S. II (33)). Thus, though the ultra-elliptic set is special, it can be used to determine the structure of the groups determined by the general set.

In §4 the relation of $g_{n,k}^{(2)}$ to modular groups determined by $e_{n,k}$ is discussed, and the restriction to self-associated sets P_{2p+2}^p is made by the use of (3).

We shall assume throughout that $n \geqq k+3$.

§1. *The Generators, Invariants and Conjugate Linear Forms of $g_{n,k}^{(2)}$.*

The group $g_{n,k}$ is generated (P. S. II §5) by transpositions such as T_{12} of the variables $x_1, \ldots, x_n$ and the involutory element

$$A_{1,2,\ldots,k+1}: \begin{array}{ll} x'_i = x_i + [(k-1)x_0 - x_1 - x_2 \ldots - x_{k+1}] & (i=0, 1, 2, \ldots, k+1), \\ x'_j = x_j \quad (j=k+1, \ldots, n). & \end{array}$$

These generators belong in a conjugate set. The group has an invariant quadric form M, and an invariant point O and linear form L which are pole and polar as to M. These are

$$\begin{aligned} M &= (k-1)x_0^2 - (x_1^2 + x_2^2 + \ldots + x_n^2), \\ L &= (k+1)x_0 - (x_1 + x_2 + \ldots + x_n), \\ O &= k+1, \ k-1, \ k-1, \ldots, k-1. \end{aligned}$$

If these generators be reduced mod. 2, and thereafter combined mod. 2, they become the generators of $g_{n,k}^{(2)}$. The above invariants similarly reduced become invariants of $g_{n,k}^{(2)}$.

Thus $g_{n,k}^{(2)}$ is generated by transpositions such as T_{12} and $A_{1,\ldots,k+1}$ where

$$(4) \quad A_{1,\ldots,k+1}: \begin{array}{ll} k \text{ even.} & k \text{ odd.} \\ x'_i = x_i + (x_0 + x_1 + \ldots + x_{k+1}), & : \ x'_i = x_i + (x_1 + \ldots + x_{k+1}), \\ x'_j = x_j, & \quad x'_j = x_j, \\ \multicolumn{2}{c}{(i=0, 1, \ldots, k+1; \ j=k+2, \ldots, n).} \end{array}$$

Writing M above in polarized form before reducing we have as invariants of $g_{n,k}^{(2)}$:

$$(5) \quad \left\{ \begin{array}{ll} k \text{ even.} & k \text{ odd.} \\ M(x,y) = x_0y_0 + x_1y_1 + \ldots + x_ny_n, & M(x,y) = x_1y_1 + \ldots + x_ny_n, \\ L = x_0 + x_1 + \ldots + x_n, & L = x_1 + \ldots + x_n, \\ O = 1, 1, \ldots, 1; & O = 0, 1, \ldots, 1. \end{array} \right.$$

It may happen that $g^{(2)}_{n,k}$ will have other quadratic invariants. Any quadratic form symmetric in $x_1, \ldots, x_n$ after possible subtraction of $M(x, x)$ takes the form

$$\alpha x_0^2+\beta x_0(x_1+\ldots+x_n)+\gamma(x_1x_2+\ldots+x_{n-1}x_n). \tag{6}$$

For k even let $x'_i=x_i+D$, $x'_j=x_j$ be $A_{1,\ldots,k+1}$. Under this element (6) acquires the increment

$$\alpha D^2+\beta\{Dx_0+D(x_1+\ldots+x_n)+D^2\}+\gamma\left\{\binom{k+1}{2}D^2+(x_{k+2}+\ldots+x_n)D\right\},$$

and this increment must vanish. After factoring out D, we find from the typical terms in x_0, x_1, x_n that $\alpha+\gamma\frac{k}{2}\equiv\beta+\gamma\equiv0$ mod. 2. Hence defining $\varkappa$ as in (2) we have when $\varkappa=0$ that $\alpha=0$, $\beta=\gamma=1$; and when $\varkappa=2$ that $\alpha=\beta=\gamma=1$. When k is odd the increment is

$$\alpha D^2+\beta D(x_1+\ldots+x_n)+\gamma\left\{\binom{k+1}{2}D^2+D^2\right\},$$

whence $\alpha+\beta+\left\{\frac{k+1}{2}+1\right\}\gamma\equiv\beta\equiv0$. If $\varkappa=1$, then $\alpha=\beta=0, \gamma=1$; and if $\varkappa=3$, then $\beta=0$, $\alpha=\gamma=1$. Hence

(7) *The* $g^{(2)}_{n,k}$ *has the additional invariant quadratic form:*

$$\begin{aligned}
\varkappa&=0, & M'&= & x_0(x_1+\ldots+x_n)&+(x_1x_2+\ldots+x_{n-1}x_n);\\
\varkappa&=2, & M'&=x_0^2+ & x_0(x_1+\ldots+x_n)&+(x_1x_2+\ldots+x_{n-1}x_n);\\
\varkappa&=1, & M'&= & &(x_1x_2+\ldots+x_{n-1}x_n);\\
\varkappa&=3, & M'&=x_0^2 & &+(x_1x_2+\ldots+x_{n-1}x_n).
\end{aligned}$$

Thus in all cases the $g^{(2)}_{n,k}$ is simply or multiply isomorphic with the group of a quadric or with a subgroup of such a group. In order to determine it more precisely we shall determine its effect upon the system of integer linear forms whose coefficients are reduced mod. 2, *i. e.*, upon the spaces S_{n-1} in the S_n of $x_0, x_1, \ldots, x_n$. Any element of $g^{(2)}_{n,k}$ is completely defined if its effect upon any $n+1$ linearly independent forms is known.

All of these forms are comprised under the following types:

$$B_{1,2,\ldots,l}=x_1+x_2+\ldots+x_l,\quad C_{1,2,\ldots,l}=x_0+x_1+x_2+\ldots+x_l. \tag{8}$$

We shall denote by $B(l)$ and $C(l)$ respectively the aggregate of forms (8) obtained by taking all sets of l variables from $x_1, \ldots, x_n$; and we shall denote further by b_1, b_2, b_3, b_4 and c_0, c_1, c_2, c_3 the aggregate of forms $B(l)$,

$C(l)$ for which $l\equiv 1, 2, 3, 4$ or $l\equiv 0, 1, 2, 3$ mod. 4 respectively. From these aggregates the invariant form

$$\begin{aligned} C_{1,2,\ldots,n} &= x_0+x_1+\ldots+x_n \quad (k \text{ even}),\\ B_{1,2,\ldots,n} &= \phantom{x_0+{}} x_1+\ldots+x_n \quad (k \text{ odd}) \end{aligned}$$

is to be excluded.

Let us consider first the

CASE I: k even.

It is then quite clear that

(9) *A form* $\frac{B(l)}{C(l)}$ *is unaltered, or is transformed into a form* $\frac{C(l+k+1-2r)}{B(l+k+1-2r)}$, *by* $A_{1,\ldots,k+1}$ *according as* $A_{1,\ldots,k+1}$ *has an* $\frac{\textit{even}}{\textit{odd}}$ *or* $\frac{\textit{odd}}{\textit{even}}$ *number* r *of subscripts in common with* $\frac{B(l)}{C(l)}$. *If the form is altered, the subscripts of the transformed form are those which belong either to* $A_{1,\ldots,k+1}$, *or to the original form, but not to both.*

When a B is transformed into a C, and this C back into a B', or a C into a B and this B back into a C', the number of subscripts is increased by $2k+2-2(r+r')$ where r, r' is odd, even or even, odd. Thus forms $B(l)$, $B(l')$ or forms $C(l)$, $C(l')$ are conjugate only when $l\equiv l'$ mod. 4. Moreover, forms $B(l)$, $C(l')$ are conjugate only when $l'\equiv l+k+1-2r$ mod. 4 (r odd). The latter condition depends upon the value of k and we find that

(10) *Under* $g^{(2)}_{n,k}$, *k even, there are four sets of conjugate linear forms, namely:*

$$\begin{aligned} \varkappa=0:\ & b_1, c_0;\ b_2, c_1;\ b_3, c_2;\ b_4, c_3;\\ \varkappa=2:\ & b_1, c_2;\ b_2, c_3;\ b_3, c_0;\ b_4, c_1; \end{aligned}$$

the linear form L being excluded.

That all of the linear forms indicated actually occur in a conjugate set is readily verified. Indeed, beginning with $B(y)$ $(y=1, 2, 3, 4)$ we obtain $C(y+k-1)$ and also, if $y=3$ or 4, $C(y+k-5)$. From these we get in turn all the admissible forms $B(y)$, $B(y+4)$, ..., $B(y+2k)$; etc.

Every linear form is paired by addition to the invariant form L with another linear form. These pairs are permuted by $g^{(2)}_{n,k}$ as entities. For k even the pairs of forms are $B(l)$, $C(n-l)$. Recalling from (2) that $n=4\mu+\nu$ $(\nu=0, 1, 2, 3)$ we have the following table of conjugate forms and their pairs:

$$
(11)\qquad \left\{\begin{array}{lcccc}
 & \varkappa=0 & & \varkappa=2 & \\
\nu=0 \left\{\begin{matrix}\\ \\ \end{matrix}\right. & \begin{matrix} b_1, c_0 \\ c_3, b_4 \end{matrix} & \begin{matrix} b_2, c_1 \\ c_2, b_3 \end{matrix} & \begin{matrix} b_1, c_2 \\ c_3, b_2 \end{matrix} & \begin{matrix} b_3, c_0 \\ c_1, b_4 \end{matrix} \\
\nu=2 \left\{\begin{matrix}\\ \\ \end{matrix}\right. & \begin{matrix} b_1, c_0 \\ c_1, b_2 \end{matrix} & \begin{matrix} b_3, c_2 \\ c_3, b_4 \end{matrix} & \begin{matrix} b_1, c_2 \\ c_1, b_4 \end{matrix} & \begin{matrix} b_2, c_3 \\ c_0, b_3 \end{matrix} \\
\nu=1 \left\{\begin{matrix}\\ \\ \end{matrix}\right. & \begin{matrix} b_2, c_1 \\ c_3, b_4 \end{matrix} & \left\{\begin{matrix} b_1 \\ c_0 \end{matrix}\right. \left\{\begin{matrix} b_3 \\ c_2 \end{matrix}\right. & \begin{matrix} b_1, c_2 \\ c_0, b_3 \end{matrix} & \left\{\begin{matrix} b_2 \\ c_3 \end{matrix}\right. \left\{\begin{matrix} b_4 \\ c_1 \end{matrix}\right. \\
\nu=3 \left\{\begin{matrix}\\ \\ \end{matrix}\right. & \begin{matrix} b_1, c_0 \\ c_2, b_3 \end{matrix} & \left\{\begin{matrix} b_2 \\ c_1 \end{matrix}\right. \left\{\begin{matrix} b_4 \\ c_3 \end{matrix}\right. & \begin{matrix} b_2, c_3 \\ c_1, b_4 \end{matrix} & \left\{\begin{matrix} b_1 \\ c_2 \end{matrix}\right. \left\{\begin{matrix} b_3 \\ c_0 \end{matrix}\right.
\end{array}\right.
$$

Within any compartment conjugate forms are in line, paired forms in column and forms both conjugate and paired are indicated by }.

Turning next to the

CASE II: k odd,

we find that

(12) *A form* $\frac{B(l)}{C(l)}$ *is unaltered, or is transformed into a form* $\frac{B(l+k+1-2r)}{C(l+k+1-2r)}$, *by* $A_{1,\ldots\ldots,k+1}$ *according as* $A_{1,\ldots\ldots,k+1}$ *has an* $\frac{\textit{even}}{\textit{odd}}$ *or* $\frac{\textit{odd}}{\textit{even}}$ *number r of subscripts in common with* $\frac{B(l)}{C(l)}$.

The subscripts of the transformed form are found by the same rule as above. We find in all six sets of conjugate forms and the table analogous to (10) is:

$$
(13)\qquad \left\{\begin{array}{l}
\varkappa=1:\ b_1;\ b_2;\ b_3;\ b_4;\ c_0,\ c_2;\ c_1,\ c_3; \\
\varkappa=3:\ b_1,\ b_3;\ b_2,\ b_4;\ c_0;\ c_1;\ c_2;\ c_3.
\end{array}\right.
$$

where, as before, the form L is excluded.

On adding the form $L=B_{1\ \ldots\ldots,n}$ the forms $B(l)$, $B(n-l)$ are paired, and also the forms $C(l)$, $C(n-l)$. Hence the table formed, as in (11), of conjugate and paired forms is as follows:

$$
(14)\qquad \left\{\begin{array}{lcccccccccccc}
 & & & \varkappa=1 & & & & & & \varkappa=3 & & & \\
\nu=0 \left\{\begin{matrix}\\ \\ \end{matrix}\right. & \begin{matrix} b_1 \\ b_3 \end{matrix} & \}b_2 & \}b_4 & \}c_0 & \}c_2 & \left\{\begin{matrix} c_1 \\ c_3 \end{matrix}\right. & \left\{\begin{matrix} b_1 \\ b_3 \end{matrix}\right. & \}b_2 & \}b_4 & \}c_0 & \}c_2 & \begin{matrix} c_1 \\ c_3 \end{matrix} \\
\nu=2 \left\{\begin{matrix}\\ \\ \end{matrix}\right. & \}b_1 & \begin{matrix} b_2 \\ b_4 \end{matrix} & \}b_3 & \left\{\begin{matrix} c_0 \\ c_2 \end{matrix}\right. & \left\{\begin{matrix} c_1 \\ c_3 \end{matrix}\right. & & \left\{\begin{matrix} b_1 \\ b_3 \end{matrix}\right. & \left\{\begin{matrix} b_2 \\ b_4 \end{matrix}\right. & \begin{matrix} c_0 \\ c_2 \end{matrix} & \}c_1 & \}c_3 & \\
\nu=1 \left\{\begin{matrix}\\ \\ \end{matrix}\right. & \begin{matrix} b_1 \\ b_4 \end{matrix} & \begin{matrix} b_2 \\ b_3 \end{matrix} & \begin{matrix} c_0, c_2 \\ c_1, c_3 \end{matrix} & & & & \begin{matrix} b_1, b_3 \\ b_2, b_4 \end{matrix} & \begin{matrix} c_0 \\ c_1 \end{matrix} & \begin{matrix} c_2 \\ c_3 \end{matrix} & & & \\
\nu=3 \left\{\begin{matrix}\\ \\ \end{matrix}\right. & \begin{matrix} b_1 \\ b_2 \end{matrix} & \begin{matrix} b_3 \\ b_4 \end{matrix} & \begin{matrix} c_0, c_2 \\ c_3, c_1 \end{matrix} & & & & \begin{matrix} b_1, b_3 \\ b_4, b_2 \end{matrix} & \begin{matrix} c_0 \\ c_3 \end{matrix} & \begin{matrix} c_2 \\ c_1 \end{matrix} & & &
\end{array}\right.
$$

In order to identify these linear forms with certain configurations of points and O, E quadrics associated [cf. F. G. I, §§ 1, 3] with a linear complex or null system C_p in an odd space S_{2p-1}, the basis notation described in the next section is very convenient.

§ 2. *The Basis Notation for C_p. Certain Subgroups of the Group GC_p.*

1°. With $2p+2$ subscripts $1, 2, \ldots, 2p+2$, the $2^{2p}-1$ points of S_{2p-1} (mod. 2) are indicated by

$$P_{12},\ P_{1234},\ P_{123456},\ \ldots,\quad \text{where } P_{1,2,\ldots,2k}=P_{2k+1,\ldots,2p+2}.$$

Furthermore, $P_{1,2,\ldots,2p+2}$ represents no point, and among the subscripts of a point like subscripts cancel. The three points

$$P_{i_1i_2\ldots i_{2k}},\quad P_{j_1j_2\ldots j_{2l}},\quad P_{i_1i_2\ldots i_{2k}j_1j_2\ldots j_{2l}}$$

are on a line. By means of this condition all linear spaces in S_{2p-1} can be constructed [F. G. II, § 1].

2°. The points $P_{i_1i_2\ldots}$ and $P_{j_1j_2\ldots}$ are syzygetic or azygetic, according as they have an even or an odd number of common subscripts. This relation between the two points is mutual and a point is syzygetic to itself. The locus of points syzygetic to a given point is an S_{2p-2} on the given point, and is the null space of the given point in the null system C_p. Each null space may be named like its null point [F. G. II, Theo. 1].

3°. The 2^{2p} quadrics Q, whose polar system coincides with the null system C_p, divide into $2^{p-1}(2^p+1)E$ quadrics each containing $2^{p-1}(2^p+1)-1$ points and $2^{p-1}(2^p-1)O$ quadrics each containing $2^{p-1}(2^p-1)-1$ points [F. G. I, § 3].

4°. In the base notation the E quadrics are denoted by

$$Q_{1,2,\ldots,p+1-4j}=Q_{p+1-4j+1,\ldots,2p+2}\qquad (j=0,1,\ldots);$$

and the O quadrics are denoted by

$$Q_{1,2,\ldots,p-1-4j}=Q_{p-1-4j+1,\ldots,2p+2}\qquad (j=0,1,\ldots).$$

Thus if p is odd there is a quadric Q without subscripts, and this is an O or E quadric according as $p\equiv 1$ or 3, mod. 4. A pair of quadrics determines a point, and a quadric and a point determines another quadric, by virtue of the relation

$$Q_{i_1i_2\ldots}+Q_{j_1j_2\ldots}+P_{i_1i_2\ldots j_1j_2\ldots}=0\quad \text{[F. G. II, Theos. 4, 6].}$$

5°. A point lies on a quadric if half the number of subscripts in a symbol for the point together with the number of subscripts common to the symbols of the point and quadric is even [F. G. II, Theo. 7].

6°. The group GC_p of the null system has the order

$$NC_p = 2^{p^2}(2^{2p}-1)(2^{2p-2}-1)\ldots\ldots(2^2-1).$$

It is generated by a conjugate set of involutions $I_{i_1 i_2 \ldots\ldots}$ each of which is associated with a $P_{i_1 i_2 \ldots\ldots}$. The $I_{i_1 i_2 \ldots\ldots}$ leaves every point syzygetic with $P_{i_1 i_2 \ldots\ldots}$ unaltered, and sends every point P azygetic with $P_{i_1 i_2 \ldots\ldots}$ into the point $P + P_{i_1 i_2 \ldots\ldots}$ [F. G. I, § 1].

7°. The group of an O or E quadric has the order

$$NE_p = 2^{p^2-p+1}(2^p-1)(2^{2p-2}-1)(2^{2p-4}-1)\ldots\ldots(2^2-1);$$
$$NO_p = 2^{p^2-p+1}(2^p+1)(2^{2p-2}-1)(2^{2p-4}-1)\ldots\ldots(2^2-1).$$

The involution $I_{i_1 i_2 \ldots\ldots}$ transforms a quadric Q into itself if $P_{i_1 i_2 \ldots\ldots}$ is *not* on Q, otherwise it transforms Q into $Q + P_{i_1 i_2 \ldots\ldots}$. The group of the quadric is generated by the involutions I attached to the points not on the quadric. This group is simply transitive on the set of points on the quadric and on the set of points not on the quadric. The points on Q added to Q furnish the remaining quadrics of the same type as Q; the points not on Q furnish the quadrics of the opposite type [F. G. I, §3, II, Theo. 7].

8°. The process of section and projection by a null space and from a null point is described in F. G. I, § 2, and used in § 5. So far as we shall need it here the process is as follows: To fix ideas let the point be P_{12}, its null space be L_{12}. Then the points on L_{12} apart from P_{12} are paired on null lines of the form

$$P_{12}, \quad P_{i_1 i_2 \ldots\ldots i_{2j}}, \quad P_{12 i_1 i_2 \ldots\ldots i_{2j}}.$$

Projected from P_{12} these lines become the $2^{2\pi}-1$ points of an $S_{2\pi-1}$ ($\pi = p-1$) whose basis notation has subscripts 3, 4, ..., $2p+2$, and the above line corresponds to the point $P'_{i_1 i_2 \ldots\ldots i_{2j}}$ in $S_{2\pi-1}$. The remaining $2^{2\pi}$ ordinary lines on P_{12} of the form

$$P_{12}, \quad P_{1 i_1 i_2 \ldots\ldots i_{2j-1}}, \quad P_{2 i_1 i_2 \ldots\ldots i_{2j-1}}$$

have no trace in $S_{2\pi-1}$. If points $P_{i_1 i_2 \ldots\ldots}$, $P_{j_1 j_2 \ldots\ldots}$ on L_{12} give rise to null lines on P_{12}, or points $P'_{i_1 i_2 \ldots\ldots}$, $P'_{j_1 j_2 \ldots\ldots}$ in $S_{2\pi-1}$, these points P' are syzygetic or azygetic according as the original points P are syzygetic or azygetic. Thus the null system C_p in S_{2p-1} determines its projection and section C'_π in $S_{2\pi-1}$.

9°. The 2^{2p-1} quadrics on P_{12} are

$$Q_{1 i_1 i_2 \ldots\ldots} = Q_{2 j_1 j_2 \ldots\ldots},$$

the sets of subscripts being complementary. These quadrics are paired under I_{12} into pairs $Q_{1 i_1 i_2 \ldots\ldots}$, $Q_{2 i_1 i_2 \ldots\ldots}$. The members of a pair have the same section

by L_{12} and the null lines on P_{12} are either generators of or tangents to both quadrics. The points of $S_{2\pi-1}$ corresponding to a common generator or a common tangent of the pair is a point on or off respectively the quadric $Q'_{i_1i_2\ldots}$ in $S_{2\pi-1}$. In this way the 2^{2p-1} quadrics on P_{12} give rise to $2^{2\pi}$ pairs or to the $2^{2\pi}$ quadric Q' in $S_{2\pi-1}$ associated with C'_π. The quadric Q' is an E or O quadric according as the original pair is a pair of E or a pair of O quadrics.

We shall now apply the above notation to derive the generators, the order, and the structure of certain subgroups of GC_p. The first group which we shall consider is that group H_{12} which leaves the point P_{12} unaltered. There being $2^{2p}-1$ points P_{12} the order of H_{12} is

$$NH_{12}=NC_p\div(2^{2p}-1)=2^{2p-1}NC_\pi \qquad (\pi=p-1).$$

The null lines on P_{12} are permuted among themselves by H_{12} according to a group GC'_π of order NC_π. For if $P_{i_1i_2\ldots}$ or $P_{12i_1i_2\ldots}$ is any point on L_{12}, then either $I_{i_1i_2\ldots}$ or $I_{12i_1i_2\ldots}$ will permute the null lines on P_{12} just as the involution $I'_{i_1i_2\ldots}$ attached to the point $P'_{i_1i_2\ldots}$ of the derived space $S_{2\pi-1}$ will permute the points of this space. But these involutions generate the group GC'_π. On the other hand the elements of H_{12} must leave the derived null system C'_π unaltered. Hence, to account for the above order of H_{12}, we must determine the elements of H_{12}, which leave P_{12} and every null line on it unaltered.

We had noted that $I_{i_1i_2\ldots}$ and $I_{12i_1i_2\ldots}$ effect the same involution on the null lines on P_{12}. Since two involutions I are permutable if their points are syzygetic, we see that the involution

$$T_{i_1i_2\ldots}=I_{i_1i_2\ldots}I_{12i_1i_2\ldots}=I_{12i_1i_2\ldots}I_{i_1i_2\ldots}$$

effects the identical permutation of the null lines. The same is true of the involution

$$S_{i_1i_2\ldots}=I_{12}T_{i_1i_2\ldots}=T_{i_1i_2\ldots}I_{12}=I_{12}I_{i_1i_2\ldots}I_{12i_1i_2\ldots}.$$

The two involutions $S_{i_1i_2\ldots}$, $T_{i_1i_2\ldots}$ are associated with a null line on P_{12}, the former symmetrically and the latter with P_{12} isolated, and may therefore be named by the point $P'_{i_1i_2\ldots}$ of $S_{2\pi-1}$. We shall therefore speak of $S_{P'}$ and $T_{P'}$ where P' is any one of the $2^{2\pi}-1$ points of $S_{2\pi-1}$. The $2(2^{2\pi}-1)$ elements $T_{P'}$, $S_{P'}$ and the two elements 1, I_{12} constitute the group $\overline{H}_{12}$, which leaves P_{12} and every null line on it unaltered.

The group $\overline{H}_{12}$ is Abelian with involutory elements and its multiplication table is:

$$I_{12}S_{P'}=T_{P'},\quad I_{12}T_{P'}=S_{P'};$$

if P', $\bar{P}'$ are syzygetic, then

$$T_{P'}T_{\bar{P}'}=S_{P'+\bar{P}'}\,,\quad T_{P'}S_{\bar{P}'}=T_{P'+\bar{P}'}\,,\quad S_{P'}S_{\bar{P}'}=S_{P'+\bar{P}'}\,;$$

if P', $\bar{P}'$ are azygetic, then

$$T_{P'}T_{\bar{P}'}=T_{P'+\bar{P}'}\,,\quad T_{P'}S_{\bar{P}'}=S_{P'+\bar{P}'}\,,\quad S_{P'}S_{\bar{P}'}=T_{P'+\bar{P}'}\,.$$

The first two of these relations follow directly from the definitions of S, T; the first in each of the next sets is deduced by verifying the effect of the two members upon the $2p-1$ linearly independent points $P_{12}, P_{13}, \ldots, P_{1,2p+2}$; the remaining ones follow directly from these by inserting factors I_{12}.

The elements $T_{P'}$, $S_{P'}$ of $\bar{H}_{12}$ will interchange or leave unaltered the points on a null line of P_{12} which corresponds to $\bar{P}'$ in $S_{2\pi-1}$ according as P' and $\bar{P}'$ are azygetic or syzygetic. We ask for elements of $\bar{H}_{12}$ which effect the identical collineation in L_{12}. Such an element would leave unaltered all the S_{2p-2}'s on P_{12}, since these are null spaces of points on L_{12}, and therefore *all* the lines on P_{12}. If then it sends the point $P_{1i_1i_2\ldots}$ outside of L_{12} into the point $P_{2i_1i_2\ldots}$ it must interchange the null spaces of these points, and therefore interchange the points in which any ordinary line on P_{12} meets these spaces. Thus 1, I_{12} are the only elements of $\bar{H}_{12}$ which leave every point on L_{12} unaltered.

The group H_{12} contains subgroups simply isomorphic with the GC'_π and in fact a conjugate set of $2^{2\pi}$ such subgroups. Let $P_{1i_1i_2\ldots}$, $P_{2i_1i_2\ldots}$ be the pair of points on any one of the $2^{2\pi}$ ordinary lines on P_{12}. Their null spaces cut L_{12} in the same $S_{2\pi-1}$ which meets every null line on P_{12} in a single point $\bar{P}$. The $2^{2\pi}-1$ involutions $I_{\bar{P}}$ generate a group isomorphic with the group GC'_π on the null lines. Moreover, the group thus generated can contain no element of $\bar{H}_{12}$ since the order of the group of P_{12} and $P_{1i_1i_2\ldots}$ is $NH_{12}\div 2^{2p-1}=NC'_\pi$. A group of this kind occurs for each one of the conjugate set of $2^{2\pi}$ ordinary lines. Incidentally we have shown that

(15) *The group which leaves two azygetic points in S_{2p-1} each unaltered, has the order NC_π, and is generated by the involutions I attached to all points syzygetic with the two given points.*

No element $T_{P'}$ or $S_{P'}$ can leave an ordinary line on P_{12} unaltered. For if it did the one or the other of the two would leave each of the two remaining points of the line unaltered, and the group of one of these points and P_{12} would have an element in common with $\bar{H}_{12}$ contrary to what has been proved above. Hence

(16) *The subgroup H_{12} of GC_p, which leaves a point P_{12} and its null space L_{12} unaltered has the order $2 \,.\, 2^{2\pi}.NC'_\pi$ $(\pi=p-1)$ and is generated by the involutions I attached to points syzygetic with P_{12}. It has an invariant subgroup $\bar{H}_{12}$ of order $2 \,.\, 2^{2\pi}$ which leaves every null line on P_{12} unaltered. The factor group of $\bar{H}_{12}$ under H_{12} is a GC_π. Also $\bar{H}_{12}$, an Abelian group, has an invariant g_2 which leaves every point on L_{12} unaltered. The factor group of g_2 under $\bar{H}_{12}$ is a regular group on the $2^{2\pi}$ ordinary lines on P_{12}. The group H_{12} has a set of $2^{2\pi}$ conjugate subgroups GC_π and is the direct product of $\bar{H}_{12}$ and any one of these subgroups.*

Let Q be a quadric on P_{12}. Its section by L_{12} and projection from P_{12} is a quadric Q' in $S_{2\pi-1}$ of the same kind. The points of Q' in $S_{2\pi-1}$ arise from the generators of Q on P_{12}; the points not on Q' arise from the tangents of Q on P_{12}. The group which leaves Q unaltered has the order $NC_p \div 2^{p-1}(2^p \pm 1)$, the upper or lower sign being used according as Q is an E or O quadric. Since P_{12} is any one of the set of $2^{p-1}(2^p \pm 1)-1$ conjugate points on Q, the order of the group which leaves Q and P_{12} unaltered is $2^{2\pi}NQ'_\pi$. Clearly Q and P_{12} are unaltered by involutions I attached to all points on L_{12} which are not on Q. Moreover, in $S_{2\pi-1}$ these generate the GQ'_π. Hence the group of Q, P_{12} must contain an invariant subgroup of order $2^{2\pi}$ which consists of elements $T_{P'}$ or $S_{P'}$. If $P_{i_1i_2\ldots\ldots}$ is not on Q, then $P_{12i_1i_2\ldots\ldots}$ is not on Q and $T_{i_1i_2\ldots\ldots}$ leaves Q, P_{12} unaltered. If $P_{i_1i_2\ldots\ldots}$ is on Q, $P_{12i_1i_2\ldots\ldots}$ also is on Q, and $S_{i_1i_2\ldots\ldots}$ leaves Q, P_{12} unaltered. That the group of P_{12}, Q is generated by the involutions I attached to points on L_{12}, but not on Q, follows first from the fact proved above that they generate the factor group GQ'_π. Secondly they evidently generate the elements $T_{P'}$ where P' is a point of $S_{2\pi-1}$ not on Q' and, since among these points P' are 2π linearly independent points, the products of these must according to the multiplication table above, give rise to an element $T_{P'}$ or $S_{P'}$ for every point P'. But as we have seen, only elements $S_{P'}$ are admissible when P' is on Q'. Now the group of Q, P_{12} is also the group of Q, $\bar{Q}$ where $\bar{Q}=Q+P_{12}$, whence

(17) *The subgroup of GC_p which leaves two quadrics Q, $\bar{Q}$ of the same type each unaltered has the order $2^{2\pi}NQ'_\pi$, and is generated by the involutions I attached to all points not on either of the two quadrics. It has an invariant Abelian subgroup which consists of the involutions $T_{P'}$ or $S_{P'}$ according as P' is not or is a point of the projected quadric Q' in $S_{2\pi-1}$. The factor group is simply isomorphic with the group of Q'.*

We see also from what has been said above that

(18) *The group $\overline{H}_{12}$, which leaves all null lines on P_{12} unaltered, has a subgroup of order $2^{2\pi}$ corresponding to each pair Q, $\overline{Q}$ of quadrics on P_{12}. This subgroup is a regular group on the ordinary lines on P_{12}.*

Two quadrics Q, $\overline{Q}$ of different kinds, such that $Q+\overline{Q}=P_{12}$, meet in their common section by L_{12}. Each is unaltered by the involution I attached to any point not on either. But all such points are on L_{12}. Any null line on P_{12} (which is not on either quadric) touches the quadrics at a common point and contains therefore a point not on either one. Hence these involutions will generate the group GC'_π on the null lines of P_{12}. Moreover, I_{12} will leave every null line and also Q, $\overline{Q}$ unaltered. Hence the order of the group of Q, $\overline{Q}$ is at least $2NC'_\pi$. The group of Q has the order $NC_p \div 2^{p-1}(2^p \pm 1)$ and it is transitive on the quadrics $\overline{Q}$ of the other kind, whence the order of the group of Q, $\overline{Q}$ is

$$NC_p \div \{2^{p-1}(2^p \pm 1)\}\{2^{p-1}(2^p \pm 1)\} = 2NC'_\pi.$$

Hence

(19) *The subgroup of GC_p, which leaves two quadrics Q, $\overline{Q}$ of opposite types, such that $Q+\overline{Q}=P_{12}$ unaltered, has the order $2NC'_\pi$ and is generated by the involutions I attached to all points not on either quadric. It has an invariant g_2, namely 1, I_{12}, whose factor group is simply isomorphic with GC'_π.*

We wish now to determine the order and nature of the subgroup $[I]$ generated by involutions $[I]$ attached to all points syzygetic with a pair of syzygetic points P_{12}, P_{34}, and therefore to P_{1234} also, excluding the generators I_{12}, I_{34}, I_{1234}. We note first from the multiplication table above that

$$I_{12} = I_{56} I_{1256} I_{78} I_{1278} I_{5678} I_{125678},$$

so that the elements excluded as generators appear in the group $[I]$. The group is an invariant subgroup of the group $H_{12,34}$, the subgroup of GC_p which leaves P_{12}, P_{34} each unaltered. The order of $H_{12,34}$ is

$$2^{2p-1} NC_\pi \div 2(2^{2\pi}-1) = 2^{4p-5} NC_{\pi'} \quad (\pi' = \pi - 1),$$

since the $2(2^{2\pi}-1)$ points P_{34} syzygetic with P_{12} are conjugate under H_{12}. If we apply Theorem (16) to the points P' of $S_{2\pi-1}$ and determine thereby the group H'_{34} in $S_{2\pi-1}$ we find that the involutions $[I]$ generate a factor group of permutations of the points P', or null lines on P_{12}, whose order is $2^{2\pi-1} NC_{\pi'}$. This factor group corresponds to the invariant subgroup of the group $[I]$

made up of all elements $T_{P'}$ or $S_{P'}$ of $\bar{H}_{12}$ which leave P_{34} unaltered and which are generated by $[I]$. The pair $T_{P'}$, $S_{P'}$ leave P_{34} unaltered only if P' is one of the $(2^{2p-4}-1)-1$ points distinct from P'_{34} and syzygetic with P'_{34}. These pairs are generated by $[I]$ as are also the four additional elements $1, I_{12}, I_{34}I_{1234}, I_{12}I_{34}I_{1234}$, whence the required invariant subgroup is of order 2^{2p-3}, and the order of the group $[I]$ is $2^{4p-6}NC_{\pi'}$ and its index under $H_{12,34}$ is 2. Hence

(20) *The group generated by the involutions $[I]$ attached to all points syzygetic with two given syzygetic points, but not on the null line of the two points has the order $2^{4p-6}NC_{\pi'}$ $(\pi'=p-2)$. It has an invariant subgroup of order 2^{4p-6}, whose factor group is $GC_{\pi'}$, which effects the identical permutation on the null planes through the line on the given points. This invariant subgroup has an invariant subgroup of order 2^{2p-3} which effects the identical permutation on the null lines through one of the given points. The group $[I]$ itself is an invariant subgroup of index 2 under the group of the two points.*

Given four quadrics Q, Q', Q'', Q''' of the same type such that $Q+Q'=P_{12}$, $Q''+Q'''=P_{12}$, $Q+Q''=P_{34}$, $Q'+Q'''=P_{34}$; we ask for the order of the group generated by the involutions $[J]$ attached to points on none of the quadrics. In $S_{2\pi-1}$ the pairs Q, Q' and Q'', Q''' become quadrics R, R'' where $R+R''=P'_{34}$, and according to Theorem (17) the group generated by $[J]$ has a factor group of order $2^{2\pi'}NQ_{\pi'}$ which corresponds to an invariant subgroup of $[J]$ consisting of those elements of $\bar{H}_{12}$ which leave each of the four quadrics unaltered. These are the identity, all elements $T_{P'}$ for which P' is not on either R, R'', and all elements $S_{P'}$ for which P' is on both R' and R''. Such points P' are all points in $S_{2\pi-1}$ syzygetic with P'_{34} including P'_{34}, whence the group has the order $2^{2\pi-1}$. Hence

(21) *The group generated by the involutions $[I]$ attached to all points not on the four quadrics which arise from any one by transforming it by involutions I attached to three points on one of its generators is of order $2^{4p-7}NQ_{\pi'}$ $(\pi'=p-2)$, where $Q_{\pi'}$ is of the same type as the given quadrics. It has an invariant subgroup of order 2^{4p-7} whose factor group is isomorphic with $GQ_{\pi'}$.*

If, on the other hand, Q'', Q''' are of a type opposite to that of Q, Q' then in $S_{2\pi-1}$, R, R'' are unlike quadrics, and we apply Theorem (19) to find the group of order $2NC_{\pi'}$ in $S_{2\pi-1}$. Here the only elements of $\bar{H}_{12}$ are $1, I_{34}I_{1234}$.

(22) *If $\overline{P_{12}, P_{34}}$ is a tangent to Q at P_{12}, the group generated by involutions I attached to all points not on the four quadrics Q, $Q+P_{12}$, $Q+P_{34}$, $Q+P_{1234}$ is of order $2^2NC_{\pi'}$ $(\pi'=p-2)$. It has an invariant subgroup made up of* 1, I_{34}, I_{1234}, $I_{34}I_{1234}$ *whose factor group is simply isomorphic with $GC_{\pi'}$.*

The groups described above include all which appear in the next section where the sixteen cases of $g^{(2)}_{n,k}$ are discussed.

§ 3. *Identification of the Group $g^{(2)}_{n,k}$.*

This identification will be effected by comparing the permutations of the forms b, c under the operations of $g^{(2)}_{n,k}$ which is generated by the transpositions such as T_{12} and by $A_{1,\ldots,k+1}$ with the permutations of certain sets of points or quadrics in the finite geometry of the null system under involutions I attached to certain points. In order that the isomorphism may be one-to-one it is necessary that the geometric objects in the conjugate set which correspond to $B_{1,\ldots,}B_n$ and C_0 be explicitly given; and secondly, that the conjugate set of forms b_2 be explicitly attached to a conjugate set of points. The first requirement ensures that under a given element the transforms of $x_1 \ldots, x_n, x_0$ can be uniquely determined; the second requirement ensures that the conjugate set of generating involutions can be determined since the transposition T_{12} has the invariant linear form B_{12}. So far as the other forms are concerned it is not necessary to identify them, but we shall usually do this sometimes explicitly, sometimes only so far as they occur in pairs.

In order to illustrate the easy passage from the permutation of the points to the algebraic transformation some invariant subgroups of $g^{(2)}_{n,k}$ are derived. The same division as in § 1 into sixteen cases is made, and in each of these the final description of the $g^{(2)}_{n,k}$ is given in the table (30) at the end of this section where also a reference to § 2 for a more complete description of the $g^{(2)}_{n,k}$ is made. The notation for the geometry is that of § 2 with subscripts $1, 2, \ldots, n$ with some additional subscripts selected from $\alpha, \beta, \gamma, \delta$.

Throughout the

CASE I: k even

we shall identify the generators as follows:

$$T_{12}=I_{12}, \quad A_{1,\ldots,k+1}=I_{1,\ldots,k+1,\alpha}.$$

For the cases $\nu=0, 1, 2, 3$ coming under $\varkappa=0$ we identify the conjugate sets of linear forms with the following sets of points in the finite geometry:

$$(23)\quad \varkappa=0\ \left\{\begin{array}{l} \nu=0, 2: n+2 \text{ subscripts } 1, \ldots, n, \alpha, \beta.\quad b_2=P_{12},\ c_1=P_{1\alpha};\\ b_1=P_{1\beta},\ b_0=P_{\alpha\beta};\quad b_3=P_{123\beta},\ c_2=P_{12\alpha\beta};\quad b_4=P_{1234},\ c_3=P_{123\alpha}.\\ \nu=1, 3: n+3 \text{ subscripts } 1, \ldots, n, \alpha, \beta, \gamma.\quad b_2=P_{12},\ c_1=P_{1\alpha};\\ b_1=P_{1\beta},\ c_0=P_{\alpha\beta};\quad b_4=P_{1234},\ c_3=P_{123\alpha};\quad b_3=P_{123\beta},\ c_2=P_{12\alpha\beta}.\end{array}\right.$$

One difference between these cases is noteworthy. When $\nu=0, 2$ paired forms are represented by the same sets of points, but this is not serious, for we need only to identify completely the conjugate sets b_2, c_1 and b_1, c_0 in order to identify completely the conjugate set of generators, and the conjugates of the reference forms. But when $\nu=1$, the forms b_1, c_0 are paired, and when $\nu=3$ the forms b_2, c_1 are paired, and in this case the forms of a pair must be separated, as they are in fact by the above notation. It is to be understood, of course, that the notation $b_2=P_{12}$ indicates

$$B_{1,2}=P_{12},\quad B_{1,3}=P_{13},\quad B_{2,3}=P_{23},\ \text{etc.};$$
$$B_{1,2,3,4,5,6}=P_{123456},\ \text{etc.};\quad B_{1,2,\ldots,10}=P_{12\ldots10};\quad \ldots\ldots$$

We have next to verify that in each of the four cases the generators T_{12}, $A_{1,\ldots,k+1}$ permute the forms just as the corresponding involutions I_{12}, $I_{1,\ldots,k+1,\alpha}$ permute the corresponding points. The rule for permuting the forms is given in § 1 (9), that for permuting the points in § 6, 6°, 2°. The consideration of a few typical instances which will be omitted here shows that the two sets of permutations are isomorphic.

The generating involutions are the involutions I attached to all points of the sets P_{12}, $P_{1,2,\ldots,k+1,\alpha}$. The various cases now divide as follows:

$\nu=0$. The generators are attached to all points not on the quadric Q_β. Here $2p+2=n+2=4\mu+2$, $p+1=2\mu+1$ and Q_β is an E or O quadric in S_{n-1} according as $\mu\equiv0, 1$, mod. 2, or $n\equiv0, 4$, mod. 8.

$\nu=2$. The generators I are attached to all points not on the quadric Q. Here $2p+2=4\mu+4$, $p+1=2\mu+2$, and Q is an O or E quadric in S_{n-1} according as $\mu\equiv0, 1$, mod. 2, or $n\equiv2, 6$, mod. 8.

$\nu=1$. The generators are attached to all points not on the quadrics Q, $Q_{\beta\gamma}$ of opposite type. Here $2p+2=4\mu+4$, $2\pi-1=4\mu-1=n-2$.

$\nu=3$. The generators are attached to all points not on the like quadrics Q_β, Q_γ. Here $2p+2=4\mu+6$, $2\pi-1=4\mu+1=n-2$.

The salient features of the group in each of these four cases are tabulated in (30) at the end of this section, and in each case a reference to § 2 is made for a more complete description of the group.

The four cases which come under $\varkappa=2$ do not differ materially from the four just considered, and it will be sufficient to set forth for them a table analogous to (23) and to indicate in each case the position of the generators I. The details for each case can then be supplied from table (30).

$$(24)\quad \varkappa=2\left\{\begin{array}{l} \nu=0,\ 2: n+2 \text{ subscripts } 1,\ \ldots,\ n,\ \alpha,\ \beta.\quad b_2=P_{12},\ c_3=P_{123\alpha};\\ b_1=P_{1\beta},\ c_2=P_{12\alpha\beta};\quad b_3=P_{123\beta},\ c_0=P_{\alpha\beta};\quad b_4=P_{1234},\ c_1=P_{1\alpha}.\\ \nu=1,\ 3: n+3 \text{ subscripts } 1,\ \ldots,\ n,\ \alpha,\ \beta,\ \gamma.\quad b_2=P_{12},\ c_3=P_{123\alpha};\\ b_1=P_{1\beta},\ c_0=P_{\alpha\beta};\quad b_4=P_{1234},\ c_1=P_{1\alpha};\quad b_3=P_{123\beta},\ c_2=P_{12\alpha\beta}. \end{array}\right.$$

The generators I are attached to all points which,

when $\nu=0$, are not on the quadric Q_α;
when $\nu=2$, are not on the quadric $Q_{\alpha\beta}$;
when $\nu=1$, are not on the like quadrics $Q_{\alpha\beta}$, $Q_{\alpha\gamma}$;
when $\nu=3$, are not on the unlike quadrics Q_α, $Q_{\alpha\beta\gamma}$.

A reference to the table shows that when $\nu=3$, $g^{(2)}_{n,k}$ has an invariant g_2; when $\nu=1$, an invariant $g_{2^{n-1}}$. As an example of the transition from the collineations in the finite geometry to the linear transformations of the variables $x_0, \ldots, x_n$ we shall derive the equations of these invariant subgroups beginning with $\nu=3$.

According to Theorem (19) the invariant g_2 is the involution $I_{\beta\gamma}$. Under $I_{\beta\gamma}$ the point $P_{1\beta}$ becomes $P_{1\gamma}=P_{2,\ldots,n,\alpha,\beta}$, and $P_{\alpha\beta}$ becomes $P_{\alpha\gamma}=P_{1,\ldots,n,\beta}$. Hence the form B_1 becomes $c_{2,3,\ldots,n}$, and the form C_0 becomes $B_{1,\ldots,n}$. But this is precisely the effect of the linear transformation

$$x'_i=x_i+L \quad (i=0,\ 1,\ \ldots,\ n;\ L=x_0+x_1+\ldots+x_n).$$

When $\nu=1$, the null lines on $P_{\beta\gamma}$, the point common to the two quadrics, are either

$$P_{\beta\gamma},\ P_{12},\ P_{3,\ldots,n,\alpha} \text{ or } P_{\beta\gamma},\ P_{1234},\ P_{5,\ldots,n,\alpha}.$$

On taking a section and projection from $P_{\beta\gamma}$ the subscripts β, γ are dropped and the pair of quadrics $Q_{\alpha\beta}$, $Q_{\alpha\gamma}$ becomes Q'_α in $S_{2\pi-1}$, while the null lines are either points P'_{12} or points P'_{1234} according as the number of subscripts $\not\equiv 0$ or $\equiv 0$, mod. 4. Points P'_{12} are not on Q'_α, points P'_{1234} are on Q'_α. According to Theorem (17) the elements of the invariant subgroup are either

$$I_{12}I_{3,\ldots,n,\alpha} \text{ or } I_{1234}I_{5,\ldots,n,\alpha}I_{\beta\gamma}.$$

Taking the first as a sample of its type we find that it transforms $P_{\alpha\beta}$, $P_{1\beta}$, $P_{3\beta}$ into $P_{3,\ldots,n,\beta}$, $P_{2\beta}$, $P_{4,\ldots,n,\alpha,\beta}$, and therefore sends C_0, B_1, B_3 into $B_{3,\ldots,n}$, B_2, $C_{4,\ldots,n}$ respectively, whence it is

$$x_0'=x_0+C_{3,\ldots,n},\quad x_i'=x_i+B_{1,2},\quad x_j'=x_j+C_{3,\ldots,n}\qquad (i=1,2;\ j=3,\ldots,n).$$

The general element of this type is obtained from this by shifting sets of four subscripts from the form C to the form B and is

$$x_0'=x_0+C,\quad x_i'=x_i+B,\quad x_j'=x_j+C$$
$$(i=1,\ldots,4c+2;\ j=4c+3,\ldots,n)$$
$$B=B_{1,2,\ldots,4c+2},\quad C=C_{4c+3,\ldots,n}.$$

Similarly we find that the general element of the other type is

$$x_0'=x_0+B,\quad x_i'=x_i+C,\quad x_j'=x_j+B$$
$$(i=1,\ldots,4c;\ j=4c+1,\ldots,n)$$
$$B=B_{1,\ldots,4c},\quad C=C_{4c+1,\ldots,n}.$$

We take up now the

Case II: k odd.

There is a sharp difference between the cases $\varkappa=1$ and $\varkappa=3$. When k is odd the generators § 1 (4) generate a group $g'^{(2)}_{n,k}$ on the variables $x_1,\ldots,x_n$ alone, and $g'^{(2)}_{n,k}$ is necessarily isomorphic with $g^{(2)}_{n,k}$. In the case $\varkappa=3$ there is according to § 1 (7), an invariant quadratic form which contains the variable x_0, and from the invariance of this form the effect of any element of $g^{(2)}_{n,k}$ upon x_0 can be found when its effect upon the variables $x_1,\ldots,x_n$ is known. Hence in this case the group $g'^{(2)}_{n,k}$ is simply isomorphic with $g^{(2)}_{n,k}$. However, in the case $\varkappa=1$ there may be an invariant subgroup of order μ of $g^{(2)}_{n,k}$ whose elements have the form

$$x_0'=x_0+F_j,\quad x_i'=x_i\qquad (i=1,\ldots,n;\ j=1,\ldots,\mu),$$

and then $g^{(2)}_{n,k}$ is in μ—1 isomorphism with $g'^{(2)}_{n,k}$.

We shall therefore consider these cases separately and begin with

Case II 1: $\varkappa=1$,

and determine first the group $g'^{(2)}_{n,k}$. According to table (14) we have now four conjugate sets of forms b_1, b_2, b_3, b_4, and the sets b_1, b_2 must be completely identified in order to determine respectively the linear transformations and the conjugate generators. We shall take for

$\nu=0, 2$: subscripts $1, 2, \ldots, n, \alpha, \beta$,
$\nu=1, 3$: subscripts $1, 2, \ldots, n, \alpha$.

We identify the generators and forms b as follows:

(25) $T_{12}=I_{12}$, $A_{1,\ldots,k+1}=I_{1,\ldots,k+1}$; $b_2=P_{12}$; $b_1=P_{1\alpha}$; $b_3=P_{123\alpha}$; $b_4=P_{1234}$.

Then from § 1 (12) we find that these linear forms are permuted by the generators precisely as the points are permuted by the involutions I, and we have only to identify the location of the generators.

$\nu=0$. The generators belong to all points not on the like quadrics Q_α, Q_β. Here $2p+2=4\mu+2$, $p+1=2\mu+1$, $2\pi-1=n-3$; and Q_α, Q_β are E or O quadrics according as $\mu\equiv 0, 1$, mod. 2.

$\nu=2$. The generators belong to all points not on the unlike quadrics Q, $Q_{\alpha\beta}$. Here $2p+2=4\mu+4$, $2\pi-1=n-3$.

$\nu=1$. The generators belong to all points not on Q_α. Here $2p+2=4\mu+2$, $p+1=2\mu+1$, and Q_α is an E or O quadric according as $\mu\equiv 0, 1$, mod. 2.

$\nu=3$. The generators belong to all points not on Q. Here $2p+2=4\mu+4$, $p+1=2\mu+2$, and Q is an E or O quadric according as $\mu\equiv 1, 0$, mod. 2.

These facts concerning $g'^{(2)}_{n,k}$ are collected in the table:

(26) $g'^{(2)}_{n,k} : \varkappa=1$

ν	Order of Invariant Subgroup	Factor Group	Invariant Quadric $n\equiv\nu'$, mod. 8	Reference
0	2^{n-2}	GQ_{n-3}	$Q=\frac{E}{O}$ if $\nu'=\frac{0}{4}$	§ 2 (17)
2	2	GC_{n-3}		§ 2 (19)
1		GQ_{n-2}	$Q=\frac{E}{O}$ if $\nu'=\frac{1}{5}$	§ 2, 7°
3		GQ_{n-2}	$Q=\frac{E}{O}$ if $\nu'=\frac{7}{3}$	§ 2, 7°

In order to obtain a representation which will separate the forms C, we will use when

$$\nu=0, 2 : \text{subscripts } 1, 2, \ldots, n, \alpha, \beta, \gamma, \delta;$$
$$\nu=1, 3 : \text{subscripts } 1, 2, \ldots, n, \alpha, \beta, \gamma;$$

and in all four of these cases will identify the generators and the forms C as follows:

(27) $T_{12}=I_{12}$, $A_{1,2,\ldots,k+1}=I_{1,2,\ldots,k+1,\alpha,\beta}$; $c_1=P_{1\alpha}$; $c_3=P_{123\beta}$; $c_0=P_{\alpha\gamma}$; $c_2=P_{12\beta\gamma}$.

Then the generators affect the forms c as is required in § 1 (9). Moreover, the behavior of the forms b can be deduced from that of the forms c for $B_1=C_0+C_1$. Taking up the four cases in order we find that when:

$\nu=0$. The generators belong to all points not on any one of the four like quadrics $Q_{\alpha\gamma}$, $Q_{\alpha\delta}$, $Q_{\beta\gamma}$, $Q_{\beta\delta}$, and the group is described in § 2 (21). Here $2p+2=4\mu+4=n+4$, $p+1=2\mu+2$ and the quadrics are of type E or O, according as $\mu\equiv 0$, 1, mod. 2.

$\nu=2$. The generators belong to all points not on any of the pairs of unlike quadrics Q_α, Q_β, $Q_{\alpha\gamma\delta}$, $Q_{\beta\gamma\delta}$. Here $2p+2=4\mu+6=n+4$.

$\nu=1$. The generators belong to all points not on the pair of like quadrics $Q_{\alpha\gamma}$, $Q_{\beta\gamma}$. Here $2p+2=4\mu+4=n+3$, $p+1=2\mu+2$ and the quadrics are E or O quadrics according as $\mu\equiv 0$, 1, mod. 2.

$\nu=3$. The generators belong to all points not on the like quadrics Q_α, Q_β. Here $2p+2=4\mu+6=n+3$, $p+1=2\mu+3$ and the quadrics are of type O, E, according as $\mu\equiv 0$, 1.

Thus a comparison of these results as listed in table (30) with the table (26) above, shows that $g^{(2)}_{n,k}$ has an invariant G_μ whose factor group is $g'^{(2)}_{n,k}$, and that $\mu=2^{n-1}$ except in the case $\nu=2$ for which $\mu=2$. When $\mu=2^{n-1}$ the subgroup G_μ must consist of elements of the form

$$x_0'=x_0+F,\ x_i'=x_i \quad (i=1,\ldots,n) \quad \text{where } F=x_{i_1}+x_{i_2}+\ldots+x_{i_{2j}} \quad (0\overline{<}2j\overline{<}n).$$

When $\mu=2$ G_μ consists of the identity and the single element $F=L$ the invariant linear form.

We take up finally the

CASE II 2 : $\varkappa=3$.

If in this case the forms c are completely identified, the transformations are determined. For all values of ν we take

(28) $$T_{12}=I_{12},\ A_{1,\ldots,k+1}=I_{1,\ldots,k+1};\ b_2=P_{12},\ b_4=P_{1234}.$$

Then for various values of ν we represent the forms c as follows:

(29)

ν	Subscripts 1, ..., n and	c_0	c_1	c_2	c_3
0	α, β	Q_α	Q_1	$Q_{12\alpha}$	Q_{123}
2	α, β	$Q_{\alpha\beta}$	$Q_{1\alpha}$	$Q_{12\alpha\beta}$	$Q_{123\alpha}$
1	α, β, γ	$Q_{\beta\gamma}$	$Q_{1\beta}$	$Q_{12\beta\gamma}$	$Q_{123\beta}$
3	α, β, γ	Q_α	$Q_{1\alpha\gamma}$	$Q_{12\alpha}$	$Q_{123\alpha\gamma}$

$\nu=0$, 2. The generators belong to all points syzygetic with $P_{\alpha\beta}$. Here $2p+2=4\mu+2=n+2$ and $S_{2\pi-1}=S_{n-3}$.

$\nu=1$, 3. The generators belong to all points syzygetic with the azygetic triad $P_{\beta\gamma}$, $P_{\gamma\alpha}$, $P_{\alpha\beta}$. Here $2p+2=4\mu+4=n+3$ and $S_{2\pi-1}=S_{n-2}$.

This completes the discussion of the sixteen cases, and the structure and orders of the various types of $g^{(2)}_{n,k}$ are readily ascertained from the table (30).

(30)

	ν	Order of Invariant Subgroup	Factor Group*	Invariant Quadric $n \equiv \nu'$, mod. 8	Reference to § 2
$\varkappa=0$	0		GQ_{n-1}	$Q={E \atop O}$ if $\nu'={0 \atop 4}$	7°
	1	2	GC_{n-2}		(19)
	2		GQ_{n-1}	$Q={E \atop O}$ if $\nu'={6 \atop 2}$	7°
	3	2^{n-1}	GQ_{n-2}	$Q={E \atop O}$ if $\nu'={7 \atop 3}$	(17)
$\varkappa=1$	0	$2^{n-2}.\,2^{n-1}$	GQ_{n-3}	$Q={E \atop O}$ if $\nu'={0 \atop 4}$	(21)
	1	2^{n-1}	GQ_{n-2}	$Q={E \atop O}$ if $\nu'={1 \atop 5}$	(17)
	2	$2\,.\,2$	GC_{n-3}		(22)
	3	2^{n-1}	GQ_{n-2}	$Q={E \atop O}$ if $\nu'={7 \atop 3}$	(17)
$\varkappa=2$	0		GQ_{n-1}	$Q={E \atop O}$ if $\nu'={0 \atop 4}$	7°
	1	2^{n-1}	GQ_{n-2}	$Q={E \atop O}$ if $\nu'={1 \atop 5}$	(17)
	2		GQ_{n-1}	$Q={E \atop O}$ if $\nu'={2 \atop 6}$	7°
	3	2	GC_{n-2}		(19)
$\varkappa=3$	0	2^{n-1}	GC_{n-3}		(16)
	1		GC_{n-2}		6°, (15)
	2	2^{n-1}	GC_{n-3}		(16)
	3		GC_{n-2}		6°, (15)

[In this table the subscript of Q and C is the dimension of the space in which the quadric or null system lies.]

It is proved in P. S. II (27) that the groups $g_{n,k}$ and $g_{n,n-k-2}$ are simply isomorphic and a transformation T which sends the one group into the other is given there. This transformation is degenerate modulo 2, so that it is not necessarily true that the modular groups $g^{(2)}_{n,k}$ and $g^{(2)}_{n,n-k-2}$ are simply isomorphic. We should expect, however, that some sort of isomorphism persists, and this is found to be the case. In the table below the distinct associated cases are opposite each other. We find from (30) that either the

* These factor groups are discussed in full in Dickson's "Linear Groups," Chapters II, VIII.

associated groups are simply isomorphic, or one has an invariant subgroup (noted on the margin) whose factor group is simply isomorphic with the other.

$$
(31)\qquad \left\{\begin{array}{lccl}
 & \varkappa,\ \nu & \varkappa'=\nu-\varkappa-2,\ \nu & \\
 & 0,\ 0 & 2,\ 0 & \\
g_2 & 0,\ 1 & 3,\ 1 & \\
 & 0,\ 3 & 1,\ 3 & \\
 & 1,\ 1 & 2,\ 1 & \\
 & 1,\ 2 & 3,\ 2 & g_{2^{n-3}} \\
g_2 & 2,\ 3 & 3,\ 3 &
\end{array}\right.
$$

We are above all interested in the sets P^p_{2p+2}, which may be required to be self-associated. The four cases in question here abstracted from (30) are:

$$
(32)\qquad \left\{\begin{array}{cccc}
p \text{ mod. } 4 & 2p+2 \text{ mod. } 8 & \text{Invariant Subgroup} & \text{Factor Group} \\
0 & 2 & & GO_{2p+1} \\
1 & 4 & G_2 4p+1 & GO_{2p-1} \\
2 & 6 & & GO_{2p+1} \\
3 & 0 & G_2 2p+1 & GC_{2p-1}
\end{array}\right.
$$

We shall show in the next section that the requirement that the set P^p_{2p+2} be self-associated does not reduce the group $g^{(2)}_{2p+2,\,p}$ when p is odd; but that when p is even this requirement reduces the group to one with an invariant G_2 whose factor group is GC_{2p-1}. Thus in all cases the self-associated set defines a modular group in the finite geometry of S_{2p-1}. In order to impose the conditions for self-association most conveniently, it is necessary to consider the ultra-elliptic set P^p_{2p+2}.

§ 4. *Modular Groups Determined by $e_{n,k}$.*

Congruent ultra-elliptic sets P^k_n, P'^k_n lie on projectively equivalent norm-elliptic curves E^{k+1}, E'^{k+1}. When E'^{k+1} is projected upon E^{k+1}, P'^k_n is projected upon a set on E^{k+1} whose elliptic parameters $u'_1, \ldots, u'_n$ are expressed in terms of the elliptic parameters $u_1, \ldots, u_n$ of P^k_n by means of a linear transformation—an element of the group $e_{n,k}$. The generators of $e_{n,k}$ are the transpositions of the u's and the element $A_{1,\ldots,k+1}$ which is (P. S. II (32))

$$
A_{1,\ldots,k+1}:\quad \begin{array}{ll}
u'_i=u_i-\dfrac{2}{k+1}(u_1+\ldots+u_{k+1}) & (i=1,\ \ldots,\ k+1), \\[2ex]
u'_j=u_j+\dfrac{k-1}{k+1}(u_1+\ldots+u_{k+1}) & (j=k+2,\ \ldots,\ n).
\end{array}
$$

All the elements of $e_{n,k}$ have for coefficients rational numbers with denominators which are factors of $k+1$ or of $\dfrac{k+1}{2}$ according as k is even or odd [P. S. II (37)].

Let us consider the effect of these elements upon linear forms,

$$x_1u_1+x_2u_2+\ldots.+x_nu_n,$$

with integer coefficients $x_1, \ldots., x_n$. This is transformed by $A_{1,\ldots.,k+1}$ into the form

$$x'_1u'_1+x'_2u'_2+\ldots.+x'_nu'_n,$$

where

$$x'_i=x_i+\frac{k-1}{k+1}\sum_1^n x_h-(x_1+\ldots.+x_{k+1}) \qquad (i=1, \ldots., k+1),$$

$$x'_j=x_j \qquad (j=k+2, \ldots., n).$$

Hence

$$\sum_1^n x'_h=\sum_1^n x_h+(k-1)\sum_1^n x_h-(k+1)(x_1+\ldots.+x_{k+1}).$$

If therefore

(33) $$\sum_{h=1}^{n} x_h=(k+1)x_0 \ (k \text{ even}), \text{ or } \sum_{h=1}^{n} x_h=\frac{k+1}{2}x_0 \ (k \text{ odd}),$$

where x_0 is an integer, then also

$$\sum_1^n x'_h=(k+1)x'_0 \ (k \text{ even}), \text{ or } \sum_1^n x'_h=\frac{k+1}{2}x'_0 \ (k \text{ odd}),$$

where x'_0 is also an integer. If we set

(34) $$y_0=x_0 \ (k \text{ even}), \text{ or } y_0=\frac{x_0}{2} \ (k \text{ odd}),$$

we find in either case that

(35) $$\begin{cases} y'_0=y_0+\{(k-1)y_0-(x_1+\ldots.+x_{k+1})\}, \\ x'_i=x_i+\{(k-1)y_0-(x_1+\ldots.+x_{k+1})\} \qquad (i=1, \ldots., k+1), \\ x'_j=x_j \qquad (j=k+2, \ldots., n). \end{cases}$$

This linear transformation (35) is precisely the generator $A_{1,\ldots.,k+1}$ of the group $g_{n,k}$. Hence

(36) *Under the group $e_{n,k}$ linear forms with integer coefficients x which satisfy the relations* (33) *are permuted contragrediently to integer linear forms under the group $g_{n,k}$.*

Hence within the totality of integer linear forms defined by (33) we shall have a modular theory identical with that of $g_{n,k}$ which has been discussed in the preceding sections for the modulus 2.

In the case of the ultra-elliptic set P^p_{2p+2} the single condition for self-association is

$$u_1+u_2+\ldots.+u_{2p+2}=0.$$

This is a linear form included in the aggregate (33). It corresponds to the point 0, 1, 1, ..., 1 for the group $g^{(2)}_{n,k}$ or through the medium of the invariant polarized form M to the linear form $B_{1,2,\ldots,n}$. When p is odd, $B_{1,\ldots,n}$ is the invariant form L so that the requirement $B_{1,\ldots,n}=0$ leads to no change in the group $g^{(2)}_{2p+2,p}$. If, however, p is even, the form $B_{1,\ldots,n}$ (see cases $\varkappa=0$, $\nu=2$; $\varkappa=2$, $\nu=2$) is represented in the finite geometry by the point $P_{\alpha\beta}$, and we ask for the subgroup of $g^{(2)}_{2p+2,p}$ which leaves $P_{\alpha\beta}$ unaltered. When $p\equiv 0$, mod. 4, there is an invariant O quadric Q not on $P_{\alpha\beta}$ so that the group reduces to that of the O, E quadrics Q, $Q_{\alpha\beta}$. When $p\equiv 2$, mod. 4, there is an invariant O quadric $Q_{\alpha\beta}$ not on $P_{\alpha\beta}$ so that the group reduces to that of the O, E quadrics $Q_{\alpha\beta}$, Q. In either case the group is reduced from a GO_{2p+1} to a group with an invariant G_2 whose factor group is a GC_{2p-1}. Now if a group H has an invariant subgroup I with factor group F, and if F has an invariant subgroup I' with factor group F', then H has a larger invariant subgroup I'' which contains I whose factor group is F'. Hence we can state that

(37) *The group $g_{2p+2,p}$ (p odd) has an invariant subgroup whose factor group is GC_{2p-1} if $p\equiv 3$, mod. 4; or GO_{2p-1} if $p\equiv 1$, mod. 4. If p is even the group $g_{2p+2,p}$ has a subgroup g' which consists of those elements which leave $x_1+\ldots+x_{2p+2}$ unaltered, mod. 2, and g' has an invariant subgroup whose factor group is GC_{2p-1}.*

The above theorem concerning $g_{2p+2,p}$ can be translated at once to apply to the simply isomorphic Cremona group $G_{2p+2,p}$ which is projectively attached to the point set P^p_{2p+2}, the subgroup g' of the theorem being that which is determined by the projective conditions for self-association. Thus the fact that the self-associated set defines groups which are isomorphic with those of the half periods of the theta functions in p variables confirms the existence of a connection between the absolute invariants of the self-associated set and the theta modular functions. In case $p\equiv 1$, mod. 4, there is indicated further that in this connection an odd theta-characteristic is isolated.

BALTIMORE, *April* 15, 1918.

On the Asymptotic Solution of the Non-Homogeneous Linear Differential Equation of the n-th Order. A Particular Solution.

By W. Van N. Garretson.

The asymptotic development* for the irregular integrals of a homogeneous linear differential equation has been obtained by both Horn† and Love.‡ Horn has published several papers on the case where the roots of the characteristic equation are all distinct, while Love has taken up the case where the roots of the characteristic equation are unrestricted as to their order of multiplicity, including the case of distinct roots as a special case.

In this paper we shall consider the non-homogeneous equation where the roots of the characteristic equation are distinct, and follow, at the outset, the method employed by Dini§ in his researches on linear differential equations. In Section I the two theorems stated and proved by him will be generalized so as to apply to the non-homogeneous equation and combined in one theorem. In Section II we shall determine a particular solution of the given equation. To this end we shall make use of the researches of Love‡ in the homogeneous linear differential equation by employing his solutions. The particular solution thus obtained of the non-homogeneous equation will be in the form of quadratures. The determination of the asymptotic development of the particular integral found in Section II will form the content of Section III.

Section I.

Take for consideration the non-homogeneous linear differential equation

$$y^{(n)}+a_1(x)y^{(n-1)}+a_2(x)y^{(n-2)}+\ldots\ldots a_n(x)y=X(x), \tag{1}$$

* "Asymptotic Development in Poincaré's Sense," cf. *Acta Mathematica*, Vol. VIII (1886), p. 297.

† *Journal für Mathematik*, Vol. CXXXVIII (1910), pp. 159-191.

‡ *Annals of Mathematics*, Second Series, Vol. XV (1914), pp. 145-156. Also American Journal of Mathematics, Vol. XXXVI, No. 2 (1914), pp. 151-166.

§ *Annali di Matematica*, Ser. 3, Vol. II (1898), pp. 297-324. *Ibid.*, Vol. III (1899), pp. 125-183. Important contributions have also been made by Poincaré, Kneser, Birkhoff, and others.

in which the coefficients are real or complex functions developable, asymptotically, for large values of x in the form

$$a_i(x) \sim x^{ik}\left[a_{i,0}+\frac{a_{i,1}}{x}+\frac{a_{i,2}}{x^2}+\dots\right]; \quad i=1,2,\dots,n; \quad k=0,1,2,\dots,$$

while the first $n-i$ derivatives also possess asymptotic developments. The function $X(x)$ will be considered as capable of asymptotic development in the form

$$X(x) \sim x^m\left[b_0+\frac{b_1}{x}+\frac{b_2}{x^2}+\dots\right],$$

where m, b_0, b_1, b_2, ..., are real or complex quantities and $b_0 \neq 0$.

We regard y for the present as a known solution of (1). Let us choose n auxiliary functions $z_1, \dots, z_n$ of x, which, with their first n derivatives, are continuous for large values of x, and such that for the same values of x the determinant

$$Q(x)=\begin{vmatrix} z_1 & z_1' & z_1'' & z_1''' \dots z_1^{(n-1)} \\ z_2 & z_2' & z_2'' & \dots\dots z_2^{(n-1)} \\ \dots & \dots & \dots & \dots \\ z_n & z_n' & z_n'' & \dots\dots z_n^{(n-1)} \end{vmatrix} \tag{2}$$

never vanishes. Place

$$\int_{\beta_i}^{x}[y^{(n)}+a_1(x_1)y^{(n-1)}+a_2(x_1)y^{(n-2)}+\dots+a_n(x_1)y]z_i(x_1)\,dx_1$$
$$=\int_{\beta_i}^{x}X(x_1)z_i(x_1)\,dx_1; \quad i=1,\dots,n \tag{3}$$

where β_i are sufficiently large, positive, real quantities, or infinity, and x takes on large real values.* It is understood that, for the interval of integration, all the functions here appearing including y and its first n derivatives are continuous.

An integration by parts gives

$$p_{i,0}y^{(n-1)}+p_{i,1}y^{(n-2)}+p_{i,2}y^{(n-3)}+\dots+p_{i,n-1}y$$
$$=\int_{\beta_i}^{x}[y(x_1)Z_i(x_1)+X(x_1)z_i(x_1)]\,dx_1+c_i, \tag{4}$$

where

$$\left.\begin{aligned} &p_{i,0}=z_i, \quad p_{i,1}=z_ia_1-z_i'=z_ia_1-p_{i,0}', \quad \dots, \quad p_{i,n-1}=z_ia_{n-1}-p_{i,n-2}', \\ &-Z_i=z_ia_n-p_{i,n-1}'=z_ia_n+\varepsilon_1(z_ia_{n-1})'+\varepsilon_2(z_ia_{n-2})''+\dots+\varepsilon_nz_i^{(n)}, \\ &c_i=[p_{i,0}y^{(n-1)}+p_{i,1}y^{(n-2)}+\dots+p_{i,n-1}y]_{x=\beta_i}; \\ &\varepsilon_i=(-1)^i; \quad i=1,\dots,n. \end{aligned}\right\} \tag{5}$$

---

*In treating the homogeneous equation, Dini takes $\beta_1=\beta_2=\dots=\beta_n$, while Horn, in a similar discussion, places, as convenience demands, $\beta_1=\beta_2=\dots=\beta_v=\infty$ and $\beta_{v+1}=\dots=\beta_n=\beta$ where $0\leq v\leq n$ and β is real and positive. The second method is the only one which seems applicable, to the problem in hand, in order to insure the convergence of all of the integrals.

For the values of x under consideration the determinant

$$P(x)=\begin{vmatrix} p_{1,0} & p_{1,1} \cdots\cdots p_{1,n-1} \\ \cdots & \cdots\cdots\cdots\cdots \\ \cdots & \cdots\cdots\cdots\cdots \\ p_{n,0} & p_{n,1} \cdots\cdots p_{n,n-1} \end{vmatrix}$$

never vanishes, because

$$P(x)=(-1)^{\frac{n(n-1)}{2}}Q(x).$$

The value of y from equations (4) is found to be

$$y(x)=g(x)+\sum_{i=1}^{n}\int_{\beta_i}^{x}(y(x_1)k_i(x,x_1)+X(x_1)K_i(x,x_1))\,dx_1, \tag{6}$$

where

$$g(x)=\frac{A(x)}{Q(x)};\quad A(x)=\begin{vmatrix} z_1 & z_1' \ldots\ldots z_1^{(n-2)} & c_1 \\ z_2 & z_2' \ldots\ldots z_2^{(n-2)} & c_2 \\ \cdots & \cdots\cdots\cdots\cdots & \cdots \\ z_n & z_n' \ldots\ldots z_n^{(n-2)} & c_n \end{vmatrix},$$

$$k_i(x,x_1)=\frac{Z_i(x_1)q_i(x)}{Q(x)}\quad\text{and}\quad K_i(x,x_1)=\frac{z_i(x_1)q_i(x)}{Q(x)},$$

$q_i(x)$ is the cofactor with respect to the *i-th* row of the last column of $Q(x)$.

The formula (6) may be used as a recursion formula. By repeated application of the recursion formula the value of y as given in (6) can be expressed in the form of an infinite series as follows:

$$y(x)=\sum_{\lambda=0}^{\infty}\mu_\lambda(x)+\tau_\lambda(x), \tag{7}$$

where

$$\mu_\lambda(x)=\sum_{i=1}^{n}\int_{\beta_i}^{x}k_i(x,x_1)\,dx_1\cdot\sum_{i=1}^{n}\int_{\beta_i}^{x_1}k_i(x_1,x_2)\,dx_2\ldots\cdot\sum_{i=1}^{n}\int_{\beta_i}^{x_{\lambda-1}}k_i(x_{\lambda-1},x_\lambda)g(x_\lambda)\,dx_\lambda,$$

$$\tau_\lambda=\sum_{i=1}^{n}\int_{\beta_i}^{x}k_i(x,x_1)\,dx_1\cdot\sum_{i=1}^{n}\int_{\beta_i}^{x_1}k_i(x_1,x_2)\,dx_2\ldots\sum_{i=1}^{n}\int_{\beta_i}^{x_{\lambda-2}}k_i(x_{\lambda-2},x_{\lambda-1})\,dx_{\lambda-1}$$
$$\cdot\sum_{i=1}^{n}\int_{\beta_i}^{x_{\lambda-1}}K_i(x_{\lambda-1}x_\lambda)X(x_\lambda)\,dx_\lambda,\qquad \lambda=1,2,3,\ldots\ldots$$

In the above $\mu_0(x)=g(x)$ and $\tau_0(x)=0$.

The expression for y as given in (7) is still regarded as a known solution of (1). The form of the solution being here obtained, we can proceed at once to state how an unknown solution could be built up. Choose n auxiliary functions $z_1, z_2, \ldots, z_n$ of x which, with their first n derivatives, are continuous for large, positive, real values of x, and such that for the same values of x the determinant $Q(x)$ as given in (2) never vanishes. The functions $Z_1, \ldots, Z_n$ of x can now be formed according to (5); also $A(x)$, $g(x)$, $k_i(x,x_1)$ and

$K_i(x, x_1)$ as given in (6). In building up $A(x)$ it should be noted that $c_1, \ldots, c_n$ are now arbitrary constants and not functions of the betas. To simplify the work choose all the c's equal to zero except c_r. The function $g(x)$ will become $g_r(x)=\frac{c_r A_r(x)}{Q(x)}$ where $A_r(x)$ is the cofactor with respect to c_r in $A(x)$. Represent the corresponding values of $\mu_\lambda(x)$ and $y(x)$ by $\mu_{\lambda,r}(x)$ and $y_r(x)$ respectively.

We shall now state and prove the following:

THEOREM I. *Suppose that a large, positive, real number β can be found such that for the values of β_i either equal to β or ∞, and for all values of x greater than β the series*

$$y_r(x)=\sum_{\lambda=0}^{\infty}(\mu_{\lambda,r}(x)+\tau_\lambda(x)) \tag{8}$$

satisfies the following conditions:

(a) The series $\Sigma\mu_{\lambda,r}(x)$ and $\Sigma\tau_\lambda(x)$ converge.

(b) The series for $y_r(x)$ when multiplied by $k_i(x, x_1)$ may be integrated term by term with respect to x_1 from β to x, or from x to ∞ in case β_i equals ∞ $i=1, \ldots, v$ where $0 \leqq v \leqq n$.

(c) The series (8) defines a function $y_r(x)$ such that each of the integrals $\int_\beta^x y_r(x)Z_i(x)dx$ and $\int_\beta^x z_i(x)X(x)dx$ (or $\int_x^\infty y_r(x)Z_i(x)dx$ and $\int_x^\infty z_i(x)X(x)dx$ in case $\beta_i=\infty$; $i=1, \ldots, v$ where $0 \leqq v \leqq n$) has a meaning when $x>\beta$.

Then for such values of x the function $y_r(x)$ is an integral of (1).

PROOF: The values of $p_{i,k}$, $i=1, \ldots, n$; $k=1, \ldots, n-1$ are given in (5). Place

$$\phi_s(x)=\int_{\beta_s}^x y_r(x)Z_s(x)dx, \qquad s=1, 2, \ldots, r-1,\ r+1, \ldots, n,$$

$$\phi_r(x)=\int_{\beta_r}^x y_r(x)Z_r(x)dx+c_r;\quad \psi_s(x)=\int_{\beta_s}^x z_s(x)X(x)dx, \qquad s=1, \ldots, n,$$

$$\Delta_r(x)=\begin{vmatrix} p_{1,0} & p_{1,1}\cdots p_{1,n-2} & \phi_1+\psi_1 \\ \cdots & \cdots & \cdots \\ p_{n,0} & p_{n,1}\cdots p_{n,n-2} & \phi_n+\psi_n \end{vmatrix};$$

$$\Delta(x)=\begin{vmatrix} p_{1,0} & p_{1,1}\cdots\cdots & p_{1,n-1} \\ \cdots & \cdots & \cdots \\ p_{n,0} & p_{n,1}\cdots\cdots & p_{n,n-1} \end{vmatrix}=(-1)^{\frac{n(n-1)}{2}}Q(x).$$

Now by condition (c) the series (8) may be written

$$y_r(x)=g_r(x)+\sum_{i=1}^{n}\int_{\beta_i}^{x}(k_i(x,x_1)y_r(x_1)+K_i(x,x_1)X(x_1))\,dx_1. \tag{9}$$

By substituting the values of $g_r(x)k_i(x,x_1)$ and $K_i(x,x_1)$ in (9), and taking account of equations (3) we find that

$$y_r(x)=\Delta_r(x)/\Delta(x),$$

so it suffices for our proof to show that this function is an integral (1).

To do this consider the system of n functions $\eta_0, \eta_1, \ldots, \eta_{n-1}$, each defined for all values of x sufficiently large by means of the following system of n linear equations:

$$p_{s,0}\eta_{n-1}+p_{s,1}\eta_{n-2}+p_{s,2}\eta_{n-3}+\ldots+p_{s,n-1}\eta_0=\phi_s+\psi_s, \qquad s=1,2,\ldots,n. \tag{10}$$

It follows that

$$\eta_0=\Delta_r(x)/\Delta(x)=y_r(x). \tag{11}$$

By differentiating (10) with respect to x and making use of (5) and (11), we find that

$$z_s(\theta-X)+p'_{s,0}\theta_1+p'_{s,1}\theta_2+p'_{s,2}\theta_3+\ldots+p'_{s,n-2}\theta_{n-1}=0, \qquad s=1,\ldots,n, \tag{12}$$

where

$$\theta=\eta'_{n-1}+a_1(x)\eta'_{n-2}+a_3(x)\eta'_{n-3}+\ldots+a_{n-1}(x)\eta'_0+a_n(x)\eta_0, \tag{13}$$

$$\theta_1=\eta_{n-1}-\eta'_{n-2};\quad \theta_2=\eta_{n-2}-\eta'_{n-3};\quad \ldots;\quad \theta_{n-1}=\eta_1-\eta'_0. \tag{14}$$

The system (13) consists of n homogeneous equations in n unknowns. Upon noting that $p'_{d,b}=z_d a_{b+1}-p_{d,b+1}\begin{cases}d=1,\ldots,n\\ b=0,1,\ldots,n-2\end{cases}$ the discriminant of the system reduces at once to $(-1)^{n-1}Q(x)$, and hence does not vanish for any values of x under consideration, whence

$$\theta-X=\theta_1=\theta_2=\ldots=\theta_{n-1}=0,$$

or, by (11) and (14),

$$\eta_s=y_r^{(s)}, \qquad s=1,2,\ldots,n.$$

Substituting these values in (13), we find

$$y_r^{(n)}+a_1(x)y_r^{(n-1)}+a_2(x)y_r^{(n-2)}+\ldots+a_n(x)y_r=X(x),$$

which was to be proved.

Section II.

In this section we shall endeavor to choose the auxiliary functions $z_1\ldots z_n$ in such a way that the above theorem may be employed to obtain a particular solution of (1). The complementary function corresponding to (1), *i. e.*, the general solution of

$$y^{(n)}+a_1(x)y^{(n-1)}+a_2(x)y^{(n-2)}+\ldots+a_n(x)y=0 \tag{15}$$

has already been obtained by Horn and Love. We shall suppose the characteristic equation of (1), viz.,

$$w^n+a_{1,0}w^{n-1}+a_{2,0}w^{n-2}+\dots+a_{n,0}=0 \tag{16}$$

to have n distinct roots $w_1, \dots, w_n$. Consider, now, the functions $Z_i(x_1)$ which occur in $k_i(x, x_1)$, $i=1, \dots, n$. These functions are all of the form of the left member of the differential equation

$$a_n z-a_{n-1}z'+a_{n-2}z''-a_{n-3}z'''+\dots+(-1)^n z^{(n)}=0, \tag{18}$$

an equation of the same type as (15). A fundamental system of solutions of (18) will be, according to the results obtained by Love,*

$$z_i(x)=e^{-f_i(x)}x^{-a_{i,0}}P_{i,0}(x), \qquad i=1, \dots, n, \tag{19}$$

where

$$f_i(x)=w_i\frac{x^{k+1}}{k+1}+\alpha_{i,-k}\frac{x^k}{k}+\alpha_{i,-k+i}\frac{x^{k-1}}{k-1}+\dots+\alpha_{i,-1}x, \tag{20}$$

and $P_{i,0}$ is of the form †

$$P_{i,0}(x)=\delta_{i,0}+\frac{\delta_{i,1}}{x}+\frac{\delta_{i,2}}{x^2}+\dots+\frac{\delta_{i,p}+\varepsilon_{i,p}(x)}{x^p}, \qquad i=1, \dots, n,$$

where p is an arbitrary positive integer and $\lim\limits_{x=\infty}\varepsilon_{i,p}(x)=0$.

By choosing the auxiliary functions as in (19), viz., the solutions of (18), the functions $Z_i(x)$ vanish, and this assures the vanishing of $k_i(x, x_1)$. The value of $y_r(x)$ as found in (8) then becomes

$$y_r(x)=g_r(x)+\sum_{i=1}^{n}\int_{\beta_i}^{x}K_i(x, x_1)X(x_1)\,dx_1. \tag{21}$$

The first term of the right member of (21), viz., $g_r(x)$, is a particular solution of the homogeneous differential equation (15). To obtain a particular solution of (1) set c_r in g_r equal to zero, thus making g_r vanish. Then we have as a particular solution of (1),

$$y(x)=\sum_{i=1}^{n}\int_{\beta_i}^{x}K_i(x, x_1)X(x_1)\,dx_1, \qquad i=1, \dots, n, \tag{22}$$

provided the conditions of Theorem I are satisfied. Of these three conditions, (a) is satisfied inasmuch as the series reduces to a single term; (b) is satisfied because the functions $k_i(x, x_1)$, $i=1, \dots, n$ vanish. It will be evident that (c) is satisfied when we obtain the asymptotic development for the right member of (22).

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* *Annals of Mathematics*, Second Series, Vol. XV (1914), pp. 145-156. Also AMERICAN JOURNAL OF MATHEMATICS, Vol. XXXVI, No. 2 (1914), pp. 151-166.

† In this paper the functions $P(x)$, generally written with subscripts, will have the form of $P_{i,0}(x)$ here given.

By taking the auxiliary functions as above stated, and replacing $X(x)$ by its value, the expression for $y(x)$ in (22) becomes

$$y(x)=\sum_{i=1}^{n}\int_{\beta_i}^{x} e^{f_i(x)}x^{a_{i,0}-(n-1)k}P_{i,1}(x)e^{-f_i(x_1)}x_1^{m-a_{i,0}}P_{i,2}(x_1)\,dx_1. \tag{23}$$

Section III.

The Asymptotic Development of $y(x)$.

Taking $y(x)$ as given in (23), the discussion of its asymptotic development will fall into two parts—Case 1 and Case 2—according to the behavior of $f_i(x)$.

Case 1.

Suppose $f_i(x)\not\equiv 0$, $i=1,\ldots,n$. We shall order the f's so that *

$$R[f_1(x)]\geqq R[f_2(x)]\geqq R[f_3(x)]\geqq\ldots\geqq R[f_r(x)]>0$$
$$\geqq R[f_{v+1}(x)]\geqq R[f_{v+2}(x)]\geqq\ldots\geqq R[f_n(x)]$$

when x is large, real, and positive. Let β_i, $i=1,\ldots,n$ be so chosen that $\beta_1=\beta_2=\ldots=\beta_v=\infty$ and $\beta_{v+1}=\beta_{v+2}=\ldots=\beta_n=\beta$, where β is to be taken sufficiently large. Then $y(x)$ in (23) becomes

$$y(x)=\sum_{i=1}^{v}S_i(x)+\sum_{i=v+1}^{n}T_i(x), \tag{24}$$

where

$$S_i(x)=-\int_{x}^{\infty} e^{f_i(x)}x^{a_{i,0}-(n-1)k}P_{i,1}(x)e^{-f_i(x_1)}x_1^{m-a_{i,0}}P_{i,2}(x_1)\,dx_1,\quad i=1,\ldots,v,$$

$$T_i(x)=\int_{\beta}^{x} e^{f_i(x)}x^{a_{i,0}-(n-1)k}P_{i,1}(x)e^{-f_i(x_1)}x_1^{m-a_{i,0}}P_{i,2}(x_1)\,dx_1,\quad i=v+1,\ldots,n.$$

Let s_i be the dominant term in S_i, $i=1,\ldots,v$. It can be written in the form

$$s_i(x)=e^{f_i(x)}x^{a_{i,0}-(n-1)k}\int_{x}^{\infty}\frac{x_1^{m-a_{i,0}}}{f'(x_1)}[-f'(x_1)]\,dx_1,\qquad i=1,\ldots,v,$$

in which $f_i'(x_1)\not\equiv 0$, since $f_i(x_1)\not\equiv 0$ for x_1 sufficiently large. An integration by parts gives

$$\begin{aligned}s_i(x)=e^{f_i(x)}x^{a_{i,0}-(n-1)k}&[\beta_{i,1,0}x_1^{m-a_{i,0}-k}\\&+\ldots+\beta_{i,1,p_{i,1}}x_1^{m-a_{i,0}-k-p_{i,1}}[1+\epsilon_{i,1}(x_1)]]e^{-f_i(x_1)}]_x^{\infty}\\&+e^{f_i(x)}x^{a_{i,0}-(n-1)k}\int_x^{\infty}(\gamma_{i,1,0}x_1^{m-a_{i,0}-k-1}\\&+\ldots+\gamma_{i,1,p_{i,1}}x_1^{m-a_{i,0}-k-p_{i,1}}[1+\epsilon_{i,2}(x_1)])e^{-f_i(x_1)}dx_1,\quad i=1,\ldots,v,\end{aligned}$$

where

$$\beta_{i,1,0}\cdot\ldots\cdot\beta_{i,1,p_{i,1}},\quad \gamma_{i,1,0}\cdot\ldots\cdot\gamma_{i,1,p_{i,1}}$$

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* By $R(a)$ we shall mean the real part of a.

are easily determined constants, and

$$\lim_{x=\infty} \epsilon_{i,1}(x) = \lim_{x=\infty} \epsilon_{i,2}(x) = 0.$$

By repeated application of integration by parts the exponents of the x's throughout are continually decreased. After l applications of integration by parts, the part still affected by the integral sign, call it $I_i(x)$, is the following:

$$I_i(x) = e^{f_i(x)} x^{\alpha_{i,0}-(n-1)k} \int_x^\infty (\gamma_{i,l,0} x_1^{m-\alpha_{i,0}-l(k+1)} + \gamma_{i,l,1} x_1^{m-\alpha_{i,0}-l(k+1)-1}$$
$$+ \dots + \gamma_{i,l,p_{i,l}} x_1^{m-\alpha_{i,0}-l(k+1)-p_{i,l}} [1+\epsilon_{i,2l}(x_1)]) e^{-f_i(x_1)} dx_1, \quad i=1, \dots, v,$$

where the γ's are easily determined constants and $\lim_{x=\infty} \epsilon_{i,2l}(x) = 0$. Suppose l is chosen so large that $R(m-\alpha_{i,0}-l(k+1)) \leqq -2$. Since $e^{-f_i(x)}$ $i=1, \dots, v$ is a monotonic decreasing function of x, a positive quantity M_i may be found such that

$$|I_i(x)| \leqq |x^{m-nk}| \frac{M_i}{x^{(k+1)(l-1)}}. \tag{27}$$

For l greater than unity the last factor here appearing may be made small at pleasure with large values of x. By referring to (25), (26) and (27), it is readily seen that s_i and consequently S_i, $i=1, \dots, v$ can be represented asymptotically in the form

$$S_i(x) \sim \left[A_{i,0} + \frac{A_{i,1}}{x} + \frac{A_{i,2}}{x^2} + \dots \right]. \tag{28}$$

Let t_i be the dominant term in T_i, $i=v+1, \dots, n$. An integration by parts gives

$$t_i = e^{f_i(x)} x^{\alpha_{i,0}-(n-1)k} (\beta_{i,1,0} x_1^{m-\alpha_{i,0}-k}$$
$$+ \dots + \beta_{i,1,p_{i,1}} x_1^{m-\alpha_{i,0}-k-p_{i,1}} [1+\epsilon_{i,1}(x_1)]) e^{-f_i(x_1)}]_\beta^x$$
$$+ e^{f_i(x)} x^{\alpha_{i,0}-(n-1)k} \int_\beta^x (\gamma_{i,1,0} x_1^{m-\alpha_{i,0}-k-1}$$
$$+ \dots + \gamma_{i,1,p_{i,1}} x_1^{m-\alpha_{i,0}-k-p_{i,1}-1} [1+\epsilon_{i,2}(x_1)]) e^{-f_i(x_1)} dx_1, \quad i=v+1, \dots, n, \tag{29}$$

where $\beta_{i,1,0}, \dots, \beta_{i,1,p_{i,1}}$ and $\gamma_{i,1,0}, \dots, \gamma_{i,1,p_{i,1}}$ are easily determined constants, and,

$$\lim_{x=\infty} \epsilon_{i,1}(x) = \lim_{x=\infty} \epsilon_{i,2}(x) = 0.$$

After l applications of the integrations by parts, the integral that still remains is as follows:

$$I_i(x) = e^{f_i(x)} x^{\alpha_{i,0}-(n-1)k} \int_\beta^x (\gamma_{i,l,0} x_1^{m-\alpha_{i,0}-l(k+1)}$$
$$+ \dots + \gamma_{i,l,p_{i,l}} x_1^{m-\alpha_{i,0}-l(k+1)-p_{i,l}} [1+\epsilon_{i,2l}(x_1)]) e^{-f_i(x_1)} dx_1,$$
$$i=v+1, \dots, n. \tag{30}$$

For large values of β, and for $x>\beta$, each term in the integrand of (30) is a monotone increasing function of x_1 in the interval of integration. A positive quantity M_i can be found such that

$$|I_i(x)|\leqq|x^{m-nk}|\frac{M_i}{x^{(l-1)(k+1)}}, \qquad i=v+1,\ldots,n. \tag{31}$$

For l greater than unity the last factor here appearing can be made small at pleasure for large values of x. From (29), (30) and (31) it follows that t_i, and consequently T_i, $i=v+1,\ldots,n$, can be developed asymptotically in the form

$$T_i(x)\sim\left[A_{i,0}+\frac{A_{i,1}}{x}+\frac{A_{i,2}}{x^2}+\ldots\right]. \tag{32}$$

Case 2.

Suppose $f_r(x)\equiv 0$ where r is one and only one of the integers $1,\ldots,n$. Only one f could be identically zero for the roots of the characteristic equation (16), are distinct.

Referring to (23), the part arising from $f_r(x)$, in the expression for $y(x)$, is as follows:

$$u(x)=x^{a_{i,0}-(n-1)k}\left(c_{r,0}+\frac{c_{r,1}}{x}+\ldots+\frac{c_{r,n}+\varepsilon_p(x)}{x^p}\right)$$
$$\int_\beta^x x_1^{m-a_{i,0}}\left(d_{r,0}+\frac{d_{r,1}}{x_1}+\ldots+\frac{d_{r,q}+\varepsilon_q(x_1)}{x_1^q}\right)dx_1,$$

where p and q are arbitrary positive integers and $\lim\limits_{x=\infty}\varepsilon_p(x)=\lim\limits_{x=\infty}\varepsilon_q(x)'=0$. It follows that

$$u(x)\sim x^{m-nk+k+1}\left[B_0+\frac{B_1}{x}+\frac{B_2}{x^2}+\ldots\right],$$

provided that $m-\alpha_{r,0}\neq s$, when s takes on the values $-1, 0, 1, \ldots, q-1$. If, however, $m-\alpha_{r,0}=s$, then the development of $u(x)$ is

$$u(x)\sim x^{m-nk+k+1}\left[B_0+\frac{B_1}{x}+\ldots\right]+x^{a_{r,0}-nk+k}\left[c_0+\frac{c_1}{x}+\frac{c_2}{x^2}+\ldots\right]\log x.$$

That part of the expansion of (23) arising from $i=1,\ldots,r-1,\ r+1,\ldots,n$ takes the same form as (28) or (32) of Case 1.

In summary we are able to state the following:

THEOREM II. *In the differential equation*

$$y^{(n)}+a_1(x)y^{(n-1)}+a_2(x)y^{(n-2)}+\ldots+a_n(x)y=X(x), \tag{1}$$

suppose that a_i and X are real or complex functions of x developable asymptotically, when x is large, real, and positive, in the forms

$$a_i(x) \sim x^{ik}\left[a_{i,0} + \frac{a_{i,1}}{x} + \frac{a_{i,2}}{x^2} + \ldots\right], \quad i=1, \ldots, n; \; k=0, 1, 2, \ldots.$$

$$X(x) \sim x^m\left[b_0 + \frac{b_1}{x} + \frac{b_2}{x^2} \ldots\ldots\right], \quad b_0 \neq 0,$$

while the first $n-i$ derivatives of $a_i(x)$ also possess asymptotic developments.

Consider the roots $w_1 \ldots . w_n$ of the characteristic equation

$$w^n + a_{1,0}w^{n-1} + a_{2,0}w^{n-2} + \ldots + a_{n,0} = 0$$

as all distinct. Let z_i represent certain determinate functions of x of the form

$$z_i(x) = e^{-f_i(x)}x^{-\alpha_{i,0}}P_i(x),$$

where

$$f_i(x) = \frac{w_i x^{k+1}}{k+1} + \alpha_{i,-k}\frac{x^k}{k} + \alpha_{i,-k+1}\frac{x^{k-1}}{k-1} + \ldots + \alpha_{i,-1}x,$$

and

$$P_i(x) \sim \left[c_{i,0} + \frac{c_{i,1}}{x} + \frac{c_{i,2}}{x^2} + \ldots\right]; \quad i=1, \ldots, n,$$

when x is large, real, and positive. Then for the same values of x there exists a particular solution, $y(x)$, of (1), *which can be developed asymptotically as follows:*

(a) $y(x) \sim x^{m-nk}\left[A_0 + \frac{A_1}{x} + \frac{A_2}{x^2} + \ldots\right]$, *if $f_i(x) \not\equiv 0$, $i=1, \ldots, n$.*

(b) $y(x) \sim x^{m-(n-1)k+1}\left[B_0 + \frac{B_1}{x} + \frac{B_2}{x^2} + \ldots\right]$, *if $f_r(x) \equiv 0$, where r is one and only one of the values* 1, 2, ..., *n, and if at the same time $m - \alpha_{r,0} \neq S$, when S takes on the values* $-1, 0, 1, 2 \ldots.$

(c) $y(x) \sim x^{m-(n-1)k+1}\left[B_0 + \frac{B_1}{x} + \frac{B_2}{x^2} + \ldots\right] + x^{\alpha_{r,0}-(n-1)k}\left[c_0 + \frac{c_1}{x} + \ldots\right]\log x,$ *if $f_r(x) \equiv 0$ where r is one and only one of the values* 1, 2, ..., *n, and if at the same time $m - \alpha_{r,0} = S$ when S is minus one, zero, or a positive integer.*

UNIVERSITY OF MICHIGAN.

A Collineation Group Isomorphic with the Group of the Double Tangents of the Plane Quartic.

By C. C. Bramble.

Introduction.

The group of the double tangents of a plane quartic is isomorphic with one of a series of groups arising in connection with the theta functions. This one is associated with the division into half-periods for $p=3$. Its immediate predecessor associated analytically with the division into half-periods for $p=2$ is the group of order $16 \cdot 720$ associated geometrically with the Kummer* surface. A similar one† is determined by the division into thirds of periods for theta functions for $p=2$, and is associated geometrically with the lines of a cubic surface. In all these cases isomorphic collineation groups have been discovered and discussed in considerable detail, but no collineation group isomorphic with the group of the double tangents has been discussed. It is the purpose of this paper to derive such a group. The group being connected with the quartic curve, by proper mapping methods a collineation group is obtained in which the variables are irrational invariants of the quartic itself. The equation of the quartic and its double tangents are obtained in a form whose symmetry and simplicity leave nothing to be desired. A complete system for the collineation group and associated canonical forms of the quartic are obtained. The collineation group appears in seven variables. That this is the smallest number of variables in which this group can be represented as a collineation group is evident from a theorem of Wiman in Weber, "Lehrbuch der Algebra," Vol. II, p. 376. The results obtained are applicable to the solution of the equation of the double tangents of the quartic, and should also be valuable for discussing certain invariants of the quartic and configurations of the double tangents. The quartic appears with an isolated flex and may throw some light on the hitherto unsolved problem of the flexes.

* An account of this group in relation to the Kummer surface is to be found in Hudson's "Kummer's Quartic Surface," which appeared in 1905.

† This group was discussed by Burkhardt, who gave an historical account of the matter up to the time of his papers (about 1890). They appeared in the *Math. Annalen*, Vols. XXXV, XXXVI, XXXVIII.

I.

The Cremona Group $G_{7,2}$ of P_7^2 in S_6.

Two sets of seven points in a plane, P_7^2 and Q_7^2, ordered with respect to each other, are *congruent under the Cremona transformation* C_m with ρ F-points if ρ of the pairs p_i, q_i $(i=1, 2, \ldots, 7)$ are corresponding F-points of C_m, and if the remaining $7-\rho \geqq 0$ of the pairs p_i, q_i are pairs of ordinary corresponding points under C_m. The number of projectively distinct sets congruent to P_7^2 is the number of types of Cremona transformations. To determine this number the following theorem* is necessary:

The general Cremona transformation C_m $(m>2)$ with ρ F-points is projectively determined when there are given the order m, the ρ F-points, their multiplicities subject to the conditions $\sum_1^\rho r_i^2=m^2-1$, $\sum_1^\rho r_i=3(m-1)$, and the positions of four corresponding F-points.

The possible transformations to be considered in connection with P_7^2 are given by the following table where α_j is the number of F-points of multiplicity j:

	C_2	C_3	C_4	D_4	C_5	D_5	D_6	D_7	D_8
α_1	3	4	3	6	0	3	1	0	0
α_2		1	3	0	6	3	4	3	0
α_3				1		1	2	4	7

C is used to indicate a transformation with six or fewer F-points, D one with seven F-points. Using in addition the collineation C_1 we find the number of transformations C_1, C_2, C_3, C_4, C_5, D_4, D_5, D_6, D_7, D_8 to be respectively $\binom{7}{0}$, $\binom{7}{3}$, $\binom{7}{4}\binom{3}{1}$, $\binom{7}{3}\binom{4}{3}$, $\binom{7}{6}$, $\binom{7}{6}$, $\binom{7}{3}\binom{4}{3}$, $\binom{7}{1}\binom{6}{4}$, $\binom{7}{3}$, $\binom{7}{0}$ or 2.288. But since P_7^2 and Q_7^2 congruent under D_8 are projective, there are only 288 projectively distinct types of congruence.

The sets P_7^2 and Q_7^2 are mapped upon the points of a space S_6 by taking them in the canonical form:

$$P_7^2\colon\ (1,0,0),\quad (0,1,0),\quad (0,0,1),\quad (1,1,1),\quad (x_i, y_i, u),\quad (i=1,2,3),$$
$$Q_7^2\colon\ (1,0,0),\quad (0,1,0),\quad (0,0,1),\quad (1,1,1),\quad (x'_j, y'_j, u),\quad (j=1,2,3),$$

and regarding $x_1, x_2, x_3, y_1, y_2, y_3, u$ as the coordinates of a point in S_6. Then, if two sets of points are congruent under a Cremona transformation in S_2, their maps in S_6 are corresponding points under a Cremona transformation

* Coble, "Point Sets and Allied Cremona Groups," Part II, *Transactions of the American Mathematical Society*, Vol. XVII, p. 348.

in S_6. The general Cremona transformation in S_2 can be expressed as a product of quadratic factors. The effect in S_6 corresponding to a quadratic transformation in S_2 is that of an involutory Cremona transformation. Moreover, any Cremona transformation of the kind considered is a product of transformations corresponding to quadratic transformations in S_2. Any quadratic transformation in S_2 with F-points at points of P_7^2 can be obtained from a single one by permutation of the points. Hence $G_{7,2}$, the Cremona group of P_7^2 in S_6, can be generated by the symmetric group of order 7! and a single transformation in S_6 corresponding to a quadratic transformation in S_2. The number of operations in $G_{7,2}$ is clearly the same as the number of types of congruence of sets P_7^2 if further it is required that P_7^2 be ordered. $G_{7,2}$ is thus seen to be of order 7!288.

II.

Point Sets on a Cuspidal Cubic.

The cuspidal cubic curve $C_1 \equiv x_2^3 - x_1 x_3^2 = 0$ is given parametrically by $x_1 = t^3$; $x_2 = t$; $x_3 = 1$, the parameter of the cusp being $t = \infty$ and that of the flex being $t = 0$. Hence given C_1, a set of seven points P_7^2 is determined by seven parameters t_i $(i = 1, 2, \ldots, 7)$. If, on the other hand, seven points only are given, they determine a net of cubics containing among them twenty-four cuspidal cubics. Thus the fact that C_1 is given is equivalent to the assumption of a single solution of the cusp equation of degree 24 of the net. P_7^2 determined in this way is general.

The condition that two points coincide is

$$t_i - t_j = 0. \tag{1}$$

The condition that three points t_i be on a line is

$$\sum^{3} t_i = 0. \tag{2}$$

The condition that six points t_i be on a conic is

$$\sum^{6} t_i = 0. \tag{3}$$

The quadratic transformation A_{123} with F-points at t_1, t_2 and t_3 sends C_1 into another cuspidal cubic C_1' whose points can be named by means of the same parameter t. C_1' can be sent by a collineation into C_1. This operation sends a point t on C_1' into a point t' of C_1. To determine the effect on the parameters we note that if

$$t_i' = t_i + \tfrac{1}{3}(t_1 + t_2 + t_3), \tag{4}$$

then

$$t_i' + t_j' + t_k' = t_i + t_j + t_k + t_1 + t_2 + t_3.$$

That is, the requirement that to three points on a line correspond three points on a conic through the F-points is satisfied. This gives the effect of the transformation on an ordinary point. It is clear that the condition that a point coincide with an F-point goes into the condition that the corresponding point be on the opposite F-line. Hence

$$t'_i-t'_3=t_i+t_1+t_2,$$

and by means of (4) we obtain the relation

$$t'_3=t_3-\tfrac{2}{3}(t_1+t_2+t_3).$$

The effect of A_{123} is then that of the collineation on the parameters given by the equations

$$\begin{matrix} t'_1=t_1-\frac{2}{3}(t_1+t_2+t_3), & t'_2=t_2-\frac{2}{3}(t_1+t_2+t_3), \\ t'_3=t_3-\frac{2}{3}(t_1+t_2+t_3), & t'_i=t_i+\frac{1}{3}(t_1+t_2+t_3), \end{matrix} \quad i=4,5,6,7,$$

where t_1, t_2 and t_3 are F-points, and t_i ordinary points.

The aggregate of operations obtained by taking products of A_{123} and permutations of $t_1, t_2, \ldots, t_7$ constitute the group $T_{7,2}$ of P_7^2 on C_1. An element of $T_{7,2}$ can be looked upon as the operation of passing from one P_7^2 on C_1 to a congruent one named by seven other values t'_i. We get in this way 288 projectively distinct sets of points on C_1 congruent in some order. Hence there are $7!\,288$ projectively distinct ordered sets. $T_{7,2}$ the collineation group on the variables $t_1, \ldots, t_7$ is of order $7!\,288$.

Only the ratios of the t's are essential since the transformation $t'_i=\mu t_i$ represents a projectivity of C_1 into itself and therefore the sets $t_1, \ldots, t_7$ and $\mu t_1, \ldots, \mu t_7$ are projective.

III.

Invariants of $T_{7,2}$.

An invariant of $T_{7,2}$ is a function of the t's unaltered by the operations of $T_{7,2}$. The condition that two points coincide is sent by A_{123} into the condition that two points coincide, or that three points be on a line; the condition that three points be on a line is sent into the condition that two points coincide, that three points be on a line, or that six points be on a conic; the condition that six points be on a conic is sent into the condition that six points be on a conic or that three points be on a line. The algebraic expressions for these conditions ((1), (2) and (3) of II) are permuted as stated, but may change sign. Hence

$$I_2=\sum_{21}(t_1-t_2)^2+\sum_{35}(t_1+t_2+t_3)^2+\sum_{7}(t_1+t_2+t_3+t_4+t_5+t_6)^2=9(3a_1^2-4a_2),$$

where a_i is a symmetric function of the t's of degree i, is an invariant of $T_{7,2}$ of degree 2. Likewise are found,

$$I_4=\sum_{21}(t_1-t_2)^4+\sum_{35}(t_1+t_2+t_3)^4+\sum_{7}(a_1-t_1)^4=3(3a_1^2-4a_2)^2,$$

$$\begin{aligned}I_6=&\sum_{21}(t_1-t_2)^6+\sum_{35}(t_1+t_2+t_3)^6+\sum_{7}(a_1-t_1)^6\\ =&27a_1^6-108a_1^4a_2+192a_1^2a_2^2-96a_2^3-72a_1^3a_3+24a_1a_2a_3-36a_3^2+72a_1^2a_4\\ &+48a_2a_4-72a_1a_5-288a_6,\end{aligned}$$

which are invariants of degrees 4 and 6, respectively. I_4 is seen to be a multiple of I_2^2.

IV.

The Quartic C^4 Arising from a Set of Seven Points.

The plane E_z of P_7^2 is mapped upon a plane E_y by the cubic curves on P_7^2. To the cubics of E_z correspond the lines of E_y. Hence to two residual base points of a pencil of cubics of the net on P_7^2 there corresponds one point of E_y. If, however, a double point of a curve of the net on P_7^2 is taken, it alone corresponds to a point of E_y, for the two variable intersections of curves of the net have coincided. Since the Jacobian is the locus of double points of the net, the correspondence between the Jacobian of the net on P_7^2 and its map in E_y is one to one. The Jacobian of the net on P_7^2 has double points at the points of P_7^2, and being of order 6, will have $6\times3-7\times2=4$ variable intersections with curves of the net. That is, the map of the Jacobian squared, since pairs of points have coincided on it, is a quartic curve C^4 in E_y. The cubics of the net with a double point map into the lines of C^4, but the twenty-one degenerate cubics P_{ij}, consisting of the line t_it_j and the conic on the remaining five points, and the seven cubics P_{0i}, with a double point at a point of P_7^2, map into the twenty-eight double tangents of C^4 in such a way that the seven P_{0i} give rise to an Aronhold set. The twenty-four cuspidal cubics of the net map into the flex tangents and the twenty-four cusps into the flexes of C^4.

The operations of $T_{7,2}$ transform P_7^2 into Q_7^2, and transform the net of cubics on P_7^2 into a net on Q_7^2. The curves P_{0i} and P_{ij} of the net on P_7^2 are transformed into the curves Q_{0i} and Q_{ij} of Q_7^2. The effect of the generating transformation A_{123} on the curves P_{0i} and P_{ij} is to send them into Q_{0i} and Q_{ij} in such a way as to bring about the following permutation of the pairs of subscripts:

$$(01,23)\,(02,31)\,(03,12)\,(45,67)\,(46,57)\,(47,56),$$

the other pairs being unaltered.

Hence, the effect of the product $A_{237}A_{457}A_{167}$ is that of the interchange of subscripts 0 and 7. $G_{7!}$ together with the transposition (07) generates a subgroup of $T_{7,2}$, the symmetric group $G_{8!}$ of the permutations of 0, 1, 2, ..., 7. By comparing the notation above for the curves P with the Hesse notation of the double tangents $[ik;\ i, k=1, \ldots, 8;\ i \neq k]$ of the quartic, it is seen at once that $G_{8!}$ and A_{123} effect the same permutations on the curves P as the subgroup* E and the substitution P_{1238}, which generate the group of the double tangents, effect on the double tangents. Since the order of $T_{7,2}$ is that of the group of the double tangents of the quartic,

$T_{7,2}$ is simply isomorphic with the group of the double tangents of the quartic.

V.

A Net of Cubics on P_7^2.

We will obtain the quartic map of the Jacobian for the net of cubics on $t_1, t_2, \ldots, t_7$ formed by taking the three following base cubics

$C_1 \equiv -x_1x_3^2 + x_2^3 = 0$, the cuspidal cubic above;

$$C_2 \equiv x_1^2x_2 - a_1x_1^2x_3 + a_2x_1x_2^2 - a_3x_1x_2x_3 + a_4x_2^3 - a_5x_2^2x_3 + a_6x_2x_3^2 - a_7x_3^3 = 0,$$

the cubic on $t_1, \ldots, t_7$ having no term in $x_1x_3^2$ and passing through the cusp of C_1;

$$C_3 \equiv 4x_1^3 + x_1^2(c_2x_2 - c_3x_3) + x_1(c_4x_2^2 - c_5x_2x_3) + c_6x_2^3 - c_7x_2^2x_3 + c_8x_2x_3^2 - c_9x_3^3 = 0,$$

the cubic on $t_1, \ldots, t_7$ having no term in $x_1x_3^2$ and touching C_1 at $t = -\frac{1}{2}a_1$. The a_i are symmetric functions of $t_1, \ldots, t_7$ of degree i; the c's are given in terms of the a's by

$$c_n = 4a_n - 4a_1a_{n-1} + a_1^2a_{n-2}.$$

VI.

The Jacobian $J[C_1C_2C_3]$ and its Map C^4.

We have now to determine the Jacobian J of C_1, C_2, and C_3, and obtain the quartic map of J^2 by means of the equations

$$x_1' = C_1, \quad x_2' = C_2, \quad x_3' = C_3.$$

$$\begin{aligned} J \equiv\ & 24x_1^5x_3 - 36a_1x_1^4x_2^2 + 48a_2x_1^4x_2x_3 + (-24a_3 + 3a_1c_2 - 3c_3)x_1^4x_3^2 \\ & + (-36a_3 - 6a_1c_2 + 6c_3)x_1^3x_2^3 + (72a_4 + 6a_2c_2 - 6c_4)x_1^3x_2^2x_3 \\ & + (-48a_5 - 3a_3c_2 - 6a_2c_3 + 6a_1c_4 + 3c_5)x_1^3x_2x_3^2 + (24a_6 + 3a_3c_3 - 3a_1c_5)x_1^3x_3^3 \\ & + (-36a_5 - 6a_3c_2 + 3a_2c_3 - 3a_1c_4 + 6c_5)x_1^2x_2^4 \\ & + (72a_6 + 12a_4c_2 + 3a_3c_3 - 3a_1c_5 - 12c_6)x_1^2x_2^3x_3 \end{aligned}$$

* "Finite Groups," Miller, Blichfeldt and Dickson, pp, 362–365.

$$
\begin{aligned}
&+(-108a_7-9a_5c_2-9a_4c_3+9a_1c_6+9c_7)x_1^2x_2^2x_3^2+(6a_6c_2+6a_5c_3-6a_1c_7-6c_8)x_1^2x_2x_3^3\\
&+(-3a_7c_2-3a_6c_3+3a_1c_8+3c_9)x_1^2x_3^4+(-6a_5c_2-3a_3c_4+3a_2c_5+6c_7)x_1x_2^5\\
&+(12a_6c_2+6a_5c_3+6a_4c_4-6a_2c_6-6a_1c_7-12c_8)x_1x_2^4x_3\\
&+(-18a_7c_2-12a_6c_3-6a_5c_4-3a_4c_5+3a_3c_6+6a_2c_7+12a_1c_8+18c_9)x_1x_2^3x_3^2\\
&+(18a_7c_3+6a_6c_4+3a_5c_5-3a_3c_7-6a_2c_8-18a_1c_9)x_1x_2^2x_3^3\\
&+(-6a_7c_4-3a_6c_5+3a_3c_8+6a_2c_9)x_1x_2x_3^4+(3a_7c_5-3a_3c_9)x_1x_3^5\\
&+(-3a_5c_4+3a_2c_7)x_2^6+(3a_6c_4+3a_5c_5-3a_3c_7-6a_2c_8)x_2^5x_3\\
&+(-9a_7c_4-6a_6c_5-3a_5c_6+3a_4c_7+6a_3c_8+9a_2c_9)x_2^4x_3^2\\
&+(9a_7c_5+6a_6c_6-6a_4c_8-9a_3c_9)x_2^3x_3^3+(-9a_7c_6-3a_6c_7+3a_5c_8+9a_4c_9)x_2^2x_3^4\\
&+(6a_7c_7-6a_5c_9)x_1x_3^5+(-3a_7c_8+3a_6c_9)x_3^6=0.
\end{aligned}
$$

$\left(\frac{J}{3}\right)^2$ expressed in terms of the cubics C_1, C_2, and C_3, *i. e.*, the map C^4 in E_y if the $C's$ are regarded as reference lines is:

$$
\begin{aligned}
\left(\frac{J}{3}\right)^2 \equiv\ &(64a_1a_2a_3a_5a_7-32a_2a_3a_6a_7-64a_1a_3^2a_4a_7+32a_1a_2^2a_6a_7\\
&-64a_1^2a_2^2a_5a_7+64a_1^2a_2a_3a_4a_7-64a_4a_7^2-64a_1a_5^2a_7+128a_1^2a_4a_5a_7\\
&-64a_1^3a_4^2a_7+64a_5a_6a_7-64a_1a_4a_6a_7+16a_2^2a_7^2+16a_3^2a_6^2+16a_1^2a_2^2a_6^2\\
&-32a_1a_2a_3a_6^2)C_1^4+(8a_1^2a_3^2a_6-8a_1^3a_2a_3a_6+80a_1^3a_4a_7-72a_1^2a_2a_3a_7\\
&+16a_1^3a_2^2a_7+16a_1^2a_6^2-16a_1^3a_5a_6+16a_1^4a_4a_6-112a_1^2a_5a_7+64a_1a_3^2a_7\\
&+64a_3a_4a_7-32a_1a_2a_4a_7+64a_7^2+64a_1a_6a_7+32a_1a_3a_4a_6-32a_1^2a_2a_4a_6\\
&-32a_2a_5a_7-32a_3a_5a_6+32a_1a_2a_5a_6)C_1^3C_2+(8a_1a_2a_3a_6-8a_3^2a_6\\
&+48a_1a_4a_7+8a_2a_3a_7-16a_1a_2^2a_7-16a_6^2+16a_1a_5a_6-16a_1^2a_4a_6-16a_5a_7)C_1^3C_3\\
&+(16a_2a_6-16a_1a_2a_5+8a_1^3a_5-2a_1^2a_3^2+8a_1a_3a_4+8a_3a_5-48a_1a_7)C_1^2C_2C_3\\
&+(a_1^4a_3^2-4a_1^5a_5-8a_1^4a_6+16a_1^2a_2a_6+16a_1^3a_2a_5-8a_1^3a_3a_4-8a_1^2a_3a_5\\
&-16a_1^3a_7+32a_1a_2a_7-32a_1a_3a_6+16a_5^2-32a_1a_4a_5+16a_1^2a_4^2-64a_3a_7)C_1^2C_2^2\\
&+(a_3^2-4a_1a_5+8a_6)C_1^2C_3^2+(a_1^6-4a_1^4a_2+8a_1^3a_3-16a_1^2a_4+32a_1a_5)C_1C_2^3\\
&+(8a_1^2a_2-3a_1^4-8a_1a_3-16a_4)C_1C_2^2C_3+(3a_1^2-4a_2)C_1C_2C_3^2-C_1C_3^3+16C_2^3C_3=0.
\end{aligned}
$$

The quartic C^4 is only projectively determined since the net of cubics is only projectively determined by the choice of seven points, that is, C_1, C_2, and C_3 can be any linearly independent cubics of the net. Moreover, any transformation sending the net of cubics into a net of cubics transforms C^4 point by point into itself. Suppose such a transformation is given by the equations $x_i=q_i(x')$. Then $C_i(x)$ goes into $C_i[q_1(x')\ldots.]\equiv C_i'(x')$. A point of C^4 is given by $y_i=C_i(x)$ where x is a point of J. But the transform of the point is $y_i'=C_i'(x')$. Since $C_i'(x')\equiv C_i[q_1(x')\ldots.]\equiv C_i(x)$, $y_i'=y_i$ and the point is unaltered. The curves P_{0i} and P_{ij} have, however, been permuted and the coefficients are those derived from the transformed Aronhold set and are in general altered in form. For, if the transformation A_{123} is applied to the set

of points $t_1, \ldots, t_7$, and to the cubics C_1, C_2, C_3 we obtain a new set of points $t'_1, \ldots, t'_7$ and three new cubics C'_1, C'_2, C'_3 whose coefficients contain not only the symmetric functions of the points $t'_1, \ldots, t'_7$, but in addition the F-points t'_1, t'_2, and t'_3. Likewise the transform of C_4 by A_{123}, that is, the map of $J^2[C'_1, C'_2, C'_3]$ by C'_1, C'_2, and C'_3 will contain the F-points t'_1, t'_2, and t'_3, besides the symmetric functions of $t'_1, \ldots, t'_7$. If, however, C_1, C_2, and C_3 were cubics covariant under A_{123}, then since J also is covariant, the coefficients of the quartic C^4 would be invariants of A_{123}, and since they are symmetric would be invariants of $T_{7,2}$. The cubics C_1, C_2, and C_3 are mapped into the reference lines of the plane E_y. Since C_1 is covariant and is mapped into the known flex tangent, a triangle of reference determined uniquely by the flex would arise from a set of cubics covariant with the cusp cubic. The problem of finding the above-mentioned invariants of $T_{7,2}$ is then reduced to that of finding the coefficients of C^4 referred to a triangle of reference covariant with the flex.

A simple way to determine such a covariant triangle is to take

(1) for x_1 the line C_1, the flex tangent;

(2) for x_3 the tangent to C^4 at the intersection of C_1 with C^4 other than the flex $(0, 0, 1)$;

(3) for x_2 the line joining $0, 0, 1$ with the intersection of x_3 with the polar line of $0, 0, 1$ as to the polar conic of $0, 1, 0$ as to C^4.

The above choice of reference lines gives the following linear transformation of the C's to the new variables x:

$$C_1 = x_1, \quad C_2 = \lambda x_1 + x_2, \quad C_3 = \mu x_1 + x_3,$$

where $48\lambda = 3a_1^4 - 8a_1^2 a_2 + 8a_1 a_3 + 16a_4$ and $16\mu = -a_1^6 + 4a_1^4 a_2 - 8a_1^3 a_3 + 16a_1^2 a_4 - 32a_1 a_5$.

The following expressions A_i, invariants of degree i of $T_{7,2}$, are such numerical multiples of the coefficients of the transform of C^4 as to remove fractional coefficients. α_{ijkl} is the coefficient of $x_i x_j x_k x_l$ where i, j, k, l are 1, 2, or 3.

$$A_2 = \alpha_{1233} = 3a_1^2 - 4a_2.$$

$$A_6 = 48\alpha_{1133} = 18a_1^6 - 72a_1^4 a_2 + 96a_1^3 a_3 + 32a_1^2 a_2^2 - 96a_1^2 a_4 - 32a_1 a_2 a_3 + 96a_1 a_5 - 64a_2 a_4 + 48a_3^2 + 384a_6.$$

$$A_8 = 48\alpha_{1123} = -27a_1^8 + 144a_1^6 a_2 - 192a_1^5 a_3 - 160a_1^4 a_2^2 + 192a_1^4 a_4 + 320a_1^3 a_2 a_3 - 192a_1^3 a_5 - 128a_1^2 a_2 a_4 - 160a_1^2 a_3^2 + 128a_1 a_3 a_4 - 2304a_1 a_7 + 768a_2 a_6 + 384a_3 a_5 - 256a_4^2.$$

$$A_{10} = 16\alpha_{1122} = 3a_1^{10} - 20a_1^8 a_2 + 32a_1^7 a_3 + 32a_1^6 a_2^2 - 32a_1^6 a_4 - 96a_1^5 a_2 a_3 + 32a_1^5 a_5 + 64a_1^4 a_2 a_4 + 80a_1^4 a_3^2 - 128a_1^4 a_6 - 128a_1^3 a_3 a_4 - 256a_1^3 a_7 + 256a_1^2 a_2 a_6 + 128a_1^2 a_3 a_5 + 512a_1 a_2 a_7 - 512a_1 a_3 a_6 - 1024a_3 a_7 + 256a_5^2.$$

$$
\begin{aligned}
A_{12}=6912\alpha_{1113}={}&-117a_1^{12}+936a_1^{10}a_2-1152a_1^9a_3-2352a_1^8a_2^2+5376a_1^7a_2a_3+4608a_1^7a_5\\
&+1536a_1^6a_2a_4-2816a_1^6a_3^2-6912a_1^6a_6-5376a_1^5a_2^2a_3-8064a_1^5a_2a_5+1152a_1^5a_3a_4\\
&-20736a_1^5a_7-4608a_1^4a_2^2a_4+8064a_1^4a_2a_3^2+34560a_1^4a_2a_6+23040a_1^4a_3a_5-2304a_1^4a_4^2\\
&+12288a_1^3a_2^2a_5+2304a_1^3a_2a_3^2+3072a_1^3a_2a_3a_4+55296a_1^3a_2a_7-10240a_1^3a_3^3\\
&-55296a_1^3a_3a_6-36864a_1^3a_4a_5-18432a_1^2a_2^2a_6-21504a_1^2a_2a_3a_5+6144a_1^2a_2a_4^2\\
&+12288a_1^2a_3^2a_4-55296a_1^2a_3a_7+101376a_1^2a_5^2-110592a_1a_2^2a_7+73728a_1a_2a_3a_6\\
&-24576a_1a_2a_4a_5-18432a_1a_3^2a_5+6144a_1a_3a_4^2+22184a_1a_4a_7-110592a_1a_5a_6\\
&+55296a_2a_3a_7+36864a_2a_4a_6-55296a_3^2a_6+13824a_3a_4a_5-8192a_4^3-110592a_5a_7\\
&-110592a_6^2.
\end{aligned}
$$

$$
\begin{aligned}
A_{14}=768\alpha_{1112}={}&27a_1^{14}-252a_1^{12}a_2+384a_1^{11}a_3+752a_1^{10}a_2^2-384a_1^{10}a_4-2176a_1^9a_2a_3\\
&+384a_1^9a_5-608a_1^8a_2^3+1792a_1^8a_2a_4+1568a_1^8a_3^2-768a_1^8a_6+2816a_1^7a_2^2a_3\\
&-1280a_1^7a_2a_5-2944a_1^7a_3a_4+768a_1^7a_7-1536a_1^6a_2^2a_4-3968a_1^6a_2a_3^2+2816a_1^6a_2a_6\\
&+2432a_1^6a_3a_5+1280a_1^6a_4^2+4608a_1^5a_2a_3a_4-2048a_1^5a_2a_7+2048a_1^5a_3^3-5120a_1^5a_3a_6\\
&-2048a_1^5a_4a_5-1024a_1^4a_2^2a_6-512a_1^4a_2a_3a_5-1024a_1^4a_2a_4^2-4096a_1^4a_3^2a_4\\
&+8192a_1^4a_3a_7+8192a_1^4a_4a_6-1536a_1^4a_5^2+4096a_1^3a_2^2a_7+2048a_1^3a_3^2a_5+2048a_1^3a_3a_4^2\\
&+16384a_1^3a_4a_7-12288a_1^3a_5a_6-30720a_1^2a_2a_3a_7-4096a_1^2a_2a_4a_6+8192a_1^2a_2a_5^2\\
&-2048a_1^2a_3^2a_6-2048a_1^2a_3a_4a_5-12288a_1^2a_5a_7+12288a_1^2a_6^2-8192a_1a_2a_4a_7\\
&+32768a_1a_3^2a_7+8192a_1a_3a_4a_6-8192a_1a_3a_5^2+49152a_1a_6a_7-24576a_2a_5a_7\\
&+16384a_3a_4a_7-24576a_3a_5a_6+8192a_4a_5^2+49152a_7^2.
\end{aligned}
$$

$$
\begin{aligned}
A_{18}=9\cdot\overline{16}^3\alpha_{1111}={}&63a_1^{18}-756a_1^{16}a_2+1152a_1^{15}a_3+3264a_1^{14}a_2^2-1152a_1^{14}a_4-9600a_1^{13}a_2a_3\\
&+1728a_1^{13}a_5-5600a_1^{12}a_2^3+8448a_1^{12}a_2a_4+7056a_1^{12}a_3^2-1152a_1^{12}a_6+24576a_1^{11}a_2^2a_3\\
&-12288a_1^{11}a_2a_5-13824a_1^{11}a_3a_4+4608a_1^{11}a_7+2816a_1^{10}a_2^4-16896a_1^{10}a_2^2a_4\\
&-34944a_1^{10}a_2a_3^2+6144a_1^{10}a_2a_6+21504a_1^{10}a_3a_5+6912a_1^{10}a_4^2-17152a_1^9a_2^3a_3\\
&+22016a_1^9a_2^2a_5+53760a_1^9a_2a_3a_4-29184a_1^9a_2a_7+17408a_1^9a_3^3-10752a_1^9a_3a_6\\
&-30720a_1^9a_4a_5+6656a_1^8a_2^3a_4+34816a_1^8a_2^2a_3^2-5120a_1^8a_2^2a_6-76288a_1^8a_2a_3a_5\\
&-21504a_1^8a_2a_4^2-46080a_1^8a_3^2a_4+52224a_1^8a_3a_7+61440a_1^8a_4a_6+20736a_1^8a_5^2\\
&-20480a_1^7a_2^2a_3a_4+106496a_1^7a_2^2a_7-31744a_1^7a_2a_3^3-8192a_1^7a_2a_3a_6+114688a_1^7a_2a_4a_5\\
&+72704a_1^7a_3^2a_5+36864a_1^7a_3a_4^2-24576a_1^7a_4a_7-73728a_1^7a_5a_6-8192a_1^6a_2^3a_6\\
&-4096a_1^6a_2^2a_3a_5-8192a_1^6a_2^2a_4^2+30720a_1^6a_2a_3^2a_4-299008a_1^6a_2a_3a_7\\
&-237568a_1^6a_2a_4a_6-61440a_1^6a_2a_5^2+11264a_1^6a_3^4+4096a_1^6a_3^2a_6-225280a_1^6a_3a_4a_5\\
&-8192a_1^6a_4^3+73728a_1^6a_6^2-212992a_1^5a_3^3a_7+229376a_1^5a_3^2a_7-4096a_1^5a_2a_3^2a_5\\
&+24576a_1^5a_2a_3a_4^2+139264a_1^5a_2a_4a_7+245760a_1^5a_2a_5a_6-12288a_1^5a_3^3a_4\\
&+229376a_1^5a_3^2a_7+385024a_1^5a_3a_4a_6+172032a_1^5a_3a_5^2+180224a_1^5a_4^2a_5\\
&+147456a_1^5a_6a_7+778240a_1^4a_2^2a_3a_7+81920a_1^4a_2^2a_4a_6-81920a_1^4a_2^2a_5^2\\
&-286720a_1^4a_2a_3^2a_6+40960a_1^4a_2a_3a_4a_5-32768a_1^4a_2a_4^3-122880a_1^4a_2a_5a_7\\
&-245760a_1^4a_2a_6^2+8192a_1^4a_3^3a_5-36864a_1^4a_3^2a_4^2-409600a_1^4a_3a_4a_7-466944a_1^4a_3a_5a_6\\
&-425984a_1^4a_4^2a_6-466944a_1^4a_4a_5^2+147456a_1^4a_7^2-327680a_1^3a_2^2a_4a_7-819200a_1^3a_2a_3^2a_7\\
&+163840a_1^3a_2a_3a_5^2-393216a_1^3a_2a_6a_7+163840a_1^3a_3^3a_6-16384a_1^3a_3^2a_4a_5
\end{aligned}
$$

$$
\begin{aligned}
&+65536a_1^3a_3a_4^3+196608a_1^3a_3a_5a_7+393216a_1^3a_3a_6^2-262144a_1^3a_4^2a_7\\
&+1572864a_1^3a_4a_5a_6+294912a_1^3a_5^3-983040a_1^2a_2^2a_5a_7+589824a_1^2a_2^2a_6^2\\
&+1572864a_1^2a_2a_3a_4a_7-393216a_1^2a_2a_3a_5a_6-131072a_1^2a_2a_4^2a_6+131072a_1^2a_2a_4a_5^2\\
&-393216a_1^2a_2a_7^2+327680a_1^2a_3^3a_7-131072a_1^2a_3^2a_4a_6-81920a_1^2a_3^2a_5^2\\
&-65536a_1^2a_3a_4^2a_5+393216a_1^2a_3a_6a_7+393216a_1^2a_4a_5a_7-393216a_1^2a_4a_6^2\\
&-1179648a_1^2a_5^2a_6+1179648a_1a_2^2a_6a_7-1179648a_1a_2a_3a_6^2-262144a_1a_2a_4^2a_7\\
&+393216a_1a_3^2a_5a_6+262144a_1a_3a_4^2a_6-131072a_1a_3a_4a_5^2+393216a_1a_3a_7^2\\
&-1572864a_1a_4a_6a_7-1179648a_1a_5^2a_7+1179648a_1a_5a_6^2+589824a_2^2a_7^2\\
&-1179648a_2a_3a_6a_7-393216a_2a_4a_5a_7+589824a_3^2a_6^2+524288a_3a_4^2a_7\\
&-393216a_3a_4a_5a_6+65536a_4^2a_5^2-1572864a_4a_7^2+2359296a_5a_6a_7.
\end{aligned}
$$

The equation of C^4 referred to the chosen reference lines covariant with the flex is

$$
\begin{aligned}
3A_{18}x_1^4&+144A_{14}x_1^3x_2+16A_{12}x_1^3x_3+6912A_{10}x_1^2x_2^2+2304A_8x_1^2x_2x_3\\
&+2304A_6x_1^2x_3^2+110592A_2x_1x_2x_3^2-110592x_1x_3^3+1769472x_2^3x_3=0.
\end{aligned}
$$

VII.

Double Tangents of C^4.

The double tangents of C^4 in E_y are of two types:

(1) The type $(0i)$ are the maps of the cubic curves P_{0i} of the net in E_x with a double point at t_i. There are seven of this type and they form an Aronhold set.

(2) The type (ij) are the maps of the cubic curves P_{ij} of the net in E_x consisting of the line t_it_j and the conic on the remaining five points. There are twenty-one of this type.

To determine the equation of the double tangent $(0i)$ the equation of P_{0i} must first be found. The curves of the net having a common tangent at t_i form a pencil whose equation is

$$
\begin{aligned}
k(x_2^3-x_1x_3^2)+x_1^3&+(a_2-a_1^2-a_1t_i-t_i^2)x_1^2x_2-(a_3-a_1a_2-a_1^2t_i-a_1t_i^2)x_1^2x_3\\
&+(a_4-a_1a_3-a_1a_2t_i-a_2t_i^2)x_1x_2^2-(a_5-a_1a_4-a_1a_3t_i-a_3t_i^2)x_1x_2x_3\\
&+(a_6-a_1a_5-a_1a_4t_i-a_4t_i^2)x_1x_3^2-(a_7-a_1a_6-a_1a_5t_i-a_5t_i^2)x_2^2x_3\\
&+(-a_1a_7-a_1a_6t_i-a_6t_i^2)x_2x_3^2-(-a_1a_7t_i-a_7t_i^2)x_3^3=0.
\end{aligned}
$$

There is a single member of this pencil with a double point at t_i. This is the curve for which k has such a value that the point t_i, when substituted in a derivative with respect to x_1 makes it vanish. This value of k is

$$k=t_i^6+a_2t_i^4+(a_1a_2-a_3)t_i^3-a_5t_i+(a_6-a_1a_5).$$

The map in E_y of the curve of the above pencil for this value of k, that is, the equation of the double tangent $(0i)$ is

$$4[t_i^6+a_2t_i^4+(a_1a_2-a_3)t_i^3-a_5t_i+(a_6-a_1a_5)]C_1-(a_1+2t_1)^2C_2+C_3=0.$$

The equation of the line t_it_j is

$$x_1+(s_2-s_1^2)x_2+s_1s_2x_3=0,$$

where s_k is the symmetric function of t_i and t_j of degree k. The equation of the conic on the remaining five points is

$$x_1^2+(\sigma_4-\sigma_1\sigma_3)x_2^2-\sigma_1\sigma_5x_3^2+(\sigma_2-\sigma_1^2)x_1x_2-(\sigma_3-\sigma_1\sigma_2)x_1x_3-(\sigma_5-\sigma_1\sigma_4)x_2x_3=0,$$

where σ_k is the symmetric function of the five points other than t_i and t_j. To obtain the equation of the double tangent (ij) in E_y we have to find the map of the product of the equations of the line and conic above. This product expressed in terms of C_1, C_2, and C_3, that is, the equation of the double tangent (ij) is, after removing numerical fractions

$$4(\sigma_1\sigma_5+s_1s_2\sigma_3-s_1s_2\sigma_1\sigma_2)C_1-(a_1-2s_1)^2C_2+C_3=0.$$

VIII.

Proof of the Completeness of the System of Invariants.

The determination of the t's depends on the separation of the double tangents of the quartic and the isolation of a single flex. The t's are then projective irrational invariants of the quartic. Any function of the t's of proper weight is therefore a projective invariant of the quartic. Hence any invariant of $T_{7,2}$ is an irrational invariant of the quartic, such that the only irrationality present is that of the flex. Such an invariant is expressible rationally and integrally in terms of the coefficients of the quartic and of the coordinates of the isolated flex. But since the quartic has for coefficients the invariants A_2, A_6, A_8, A_{10}, A_{12}, A_{14}, and A_{18}, and the coordinates of the flex are 0, 0, 1, every invariant of $T_{7,2}$ is rationally and integrally expressible in terms of the invariants A_2, A_6, A_8, A_{10}, A_{12}, A_{14}, and A_{18}. Moreover, it is obvious when special values are given to $a_1, a_2, \ldots, a_7$ that no one of the invariants A_i can be expressed rationally and integrally in terms of the others. Hence none of them is superfluous and

The invariants A_2, A_6, A_8, A_{10}, A_{12}, A_{14}, and A_{18} form a complete system for the group $T_{7,2}$.

IX.

The Jacobian of $A_2, A_4, \ldots, A_{18}$.

If an expression is alternating under operations of $T_{7,2}$ it contains as a factor $t_i - t_j$ and all its conjugate values under the operations of $T_{7,2}$. Hence it has as a factor

$$J \equiv \prod^{21}(t_i - t_j)\prod^{35}(t_i + t_j + t_k)\prod^{7}(a_1 - t_i),$$

which is an alternating expression of degree 63.

The Jacobian of $A_2, A_4, \ldots, A_{18}$ is an alternating expression of degree 63 and is therefore to within a numerical factor the product J. Since J at most changes sign when the operations of $T_{7,2}$ are carried out, its square is an invariant of degree 126 and is rationally expressible in terms of the invariants $A_2, A_4, \ldots, A_{18}$.

X.

The Group $T_{8,3}$ of P_8^3 a Set of Base-Points of a Net of Quadrics.

There are in general an infinite number of projectively distinct sets of eight points in space congruent to a single set P_8^3 under a Cremona transformation which can be decomposed into a product of cubic Cremona transformations with F-points at points of P_8^3. If, however, P_8^3 is a set of base-points of a net of quadrics we can make use of the following theorem:

*If P_8^3 is a set of base-points of a net of quadrics, there are only thirty-six projectively distinct sets congruent in some order to P_8^3.**

There are then thirty-six types of congruence if no account is taken of the order of the points. If we require that P_8^3 be an ordered set we have, since P_8^3 can be ordered in $8!$ ways, $8!36$ types of congruence. The aggregate of operations transforming P_8^3 into the $8!36$ congruent sets constitute a group which we will call $T_{8,3}$. Any one of these operations is, as presupposed, the product of cubic transformations which can be obtained from a single one by a permutation of the points of P_8^3. Hence $T_{8,3}$ is generated by a cubic transformation and the symmetric group of permutations of the points of P_8^3 of order $8!$. Abstractly then $T_{8,3}$ has as generators precisely the set to which the generators of $T_{7,2}$ were shown to be equivalent in IV. $T_{8,3}$ and $T_{7,2}$ are therefore abstractly the same group.

* Coble, "Point Sets and Allied Cremona Groups," Part II, *Trans. Am. Math. Soc.*, Vol. XVII, p. 377 (45).

XI.

P_8^3 *on a Cuspidal Quartic.*

A cuspidal quartic curve D in space is determined by the parametric equations

$$x_1=t^4, \quad x_2=t^2, \quad x_3=t, \quad x_4=1.$$

The condition that two points t_i and t_j coincide is $t_i-t_j=0$.

The condition that four points be on a plane is $\overset{4}{\Sigma} t_i=0$.

The condition that eight points be on a quadric is $\overset{8}{\Sigma} t_i=0$.

XII.

A Net of Quadrics on P_8^3. *Generators of* $T_{8,3}$.

Since we wish to consider P_8^3 as the set of base-points of a net of quadrics, we determine P_8^3 by a choice of eight values t_i subject to the single condition $\overset{8}{\Sigma} t_i=0$, for the quartic above is the intersection of the quadrics

$$Q_2\equiv x_1x_4-x_2^2=0 \quad \text{and} \quad Q_3\equiv x_2x_4-x_3^2=0.$$

A third quadric on P_8^3 is

$$Q_1\equiv x_1^2+b_2x_1x_2-b_3x_1x_3+b_4x_2^2-b_5x_2x_3+b_6x_3^2-b_7x_3x_4+b_8x_4^2=0,$$

where b_i is the symmetric function of the t's of degree i. Hence:

P_8^3 *determined by the choice of eight t's subject to the single condition* $b_1=0$ *is the set of base-points of the net of quadrics*

$$y_1Q_1+y_2Q_2+y_3Q_3=0.$$

Generators of the group $T_{8,3}$ of P_8^3 determined in this way consist of the symmetric group of permutations of the eight t's together with a transformation on the t's corresponding to a cubic Cremona transformation A_{1234} with F-points at points of P_8^3, say at t_1, t_2, t_3, t_4; for the effect of A_{1234} is to send the net of quadrics into a net of quadrics, and the cuspidal quartic D into another cuspidal quartic D' whose points are named by the same parameter t. The quartic D' can be sent back into D by means of a collineation carrying the point t of D' into the point t' of D. Thus, can the transformation A_{1234} be regarded as a transformation upon the parameters t to new parameters t'.

The transformation

$$t_i'=t_i-\tfrac{1}{2}(t_1+t_2+t_3+t_4) \qquad (i=1,2,3,4),$$
$$t_j'=t_j+\tfrac{1}{2}(t_1+t_2+t_3+t_4) \qquad (j=5,6,7,8),$$

is seen to have the effect of permuting the conditions that two points coincide, that four points be on a plane as does the transformation A_{1234}, and thus gives the effect of A_{1234} upon P_8^3 in terms of the parameters t_i.

XIII.

The Quartic D^4.

If we consider y_1, y_2, y_3 as the coordinates of a point in a plane, we have by means of the net of quadrics

$$y_1Q_1+y_2Q_2+y_3Q_3=0,$$

a correspondence between the points y of a plane and the quadrics (yQ) of the net. To a pencil of quadrics or the elliptic quartic curve carrying the pencil corresponds a line of the plane. Corresponding to the quadrics of the net with a double point we will have a certain locus of points in the plane. Since in each pencil of the net are four quadrics with a double point, the locus is a quartic curve. Its equation in variables y_i found by writing the discriminant of the net is

$$D^4 \equiv \begin{vmatrix} 2y_1 & b_2y_1 & -b_3y_1 & y_2 \\ b_2y_1 & 2(b_4y_1-y_2) & -b_5y_1 & y_3 \\ -b_3y_1 & -b_5y_1 & 2(b_6y_1-y_3) & -b_7y_1 \\ y_2 & y_3 & -b_7y_1 & 2b_8y_1 \end{vmatrix}$$

$$\begin{aligned} &\equiv(-4b_2^2b_6b_8+b_2^2b_7^2+4b_2b_3b_5b_8-4b_3^2b_4b_8+16b_4b_6b_8-4b_4b_7^2)y_1^4 \\ &\quad+(-2b_2b_5b_7+4b_3^2b_8+4b_3b_4b_7-16b_6b_8+4b_7^2)y_1^3y_2 \\ &\quad+(4b_2^2b_8-2b_2b_3b_7-16b_4b_8+4b_5b_7)y_1^3y_3+(-4b_3b_7-4b_4b_6+b_5^2)y_1^2y_2^2 \\ &\quad+(4b_2b_6-2b_3b_5+16b_8)y_1^2y_2y_3+(b_3^2-4b_6)y_1^2y_3^2+4b_6y_1y_2^3 \\ &\quad+4b_4y_1y_2^2y_3-4b_2y_1y_2y_3^2+4y_1y_3^3-4y_2^3y_3=0. \end{aligned}$$

XIV.

Complete System for $T_{8,3}$.

The quartic D^4 consists of precisely the same terms as does C^4 above. The flex tangent now is the map of the quartic D which is unaltered by the operations of $T_{8,3}$. If we choose as base quadrics of the net quadrics covariant with D, since the discriminant is an invariant of the net, its coefficients will be invariants of $T_{8,3}$. Since the quartic D maps into the flex tangent, we need not determine these quadrics, but have only to choose in the plane a triangle of reference covariant with the flex. The lines are chosen in the same way as for C^4. The transformation is therefore of the same type as (5), removes the same terms from D^4 as (5) does from C^4 and is

$$y_1=3y_1', \quad y_2=b_4y_1'+3y_2', \quad y_3=3b_6y_1'+3y_3'.$$

The transformed expression for D^4 is

$$3B_{18}y_1'^4+27B_{14}y_1'^3y_2'+3B_{12}y_1'^3y_3'+81B_{10}y_1'^2y_2'^2+27B_8y_1'^2y_2'y_3'$$
$$+27B_6y_1'^2y_3'^2+81B_2y_1'y_2'y_3'^2+324y_1'y_3'^3-324y_2'^3y_3'=0,$$

where B_i is an invariant of $T_{8,3}$ of degree i, whose explicit expressions are as follows:

$$B_2=-4b_2,$$
$$B_6=-4b_2b_4+3b_3^2+24b_6,$$
$$B_8=-12b_2b_6-6b_3b_5+4b_4^2+48b_8,$$
$$B_{10}=-4b_3b_7+b_5^2,$$
$$B_{12}=\quad 54b_3^2b_6-18b_3b_4b_5+8b_4^3+108b_6^2,$$
$$B_{14}=-6b_2b_5b_7-6b_3b_5b_6+12b_3^2b_8+4b_3b_4b_7+2b_4b_5^2+12b_7^2,$$
$$B_{18}=\quad 27b_2^2b_7^2+108b_2b_3b_5b_8-54b_2b_3b_6b_7-18b_2b_4b_5b_7-72b_3^2b_4b_8+27b_3^2b_6^2$$
$$+24b_3b_4^2b_7-18b_3b_4b_5b_6+3b_4^2b_5^2-72b_4b_7^2-108b_5^2b_8+108b_5b_6b_7.$$

It is to be noted that great simplicity is gained in the complete system when the group is represented in eight variables whose sum is zero. This is to be expected since every term in which b_1 enters vanishes. Moreover, the notation for the double tangents is symmetrical.

The lines of D^4 arise from the pencils of quadrics of the net. The quadrics in a pencil with a double point correspond to the meets of the line with D^4. To a pencil of quadrics such that the double points have coincided in pairs corresponds a double tangent of D^4. Such a pencil of the net can be found by requiring that it contain the line t_it_j. To do this we have only to require that the quadric contain a point of the line t_it_j other than t_i and t_j. That is, the linear condition on the y''s is

$$(3\sigma_6+3s_2\sigma_4+3s_1s_2\sigma_3+3s_1^2s_2\sigma_2-s_1^2b_4-3b_6)\,y_1'-3s_1^2y_2'-3y_3'=0,$$

where s_k is the symmetric function of t_i and t_j of degree k, and σ_k the symmetric function of the remaining six t's. This is the condition in the plane that the point y be on a double tangent, that is, it is the equation of the double tangent corresponding to a choice of two of the eight points of P_8^3. The twenty-eight double tangents are thus all accounted for and are all of one type (ij).

Proof of Pohlke's Theorem and its Generalizations by Affinity.

BY ARNOLD EMCH.

1. *Introduction.*

The purpose of this paper is to show how Pohlke's Theorem, its generalizations, and some related propositions may be proved in a comprehensive manner by making use of affine collineations in space.

The theorem was first published without proof in the first part of Pohlke's "Descriptive Geometry" in 1860, and may be stated as follows:

Three straight line segments of arbitrary length in a plane, drawn from a point and making arbitrary angles with each other, form a parallel projection of three equal segments drawn from the origin on three rectangular coordinate-axes; however, only one of the segments, or one of the angles can vanish.

The first elementary rigorous proof of the fundamental theorem of axonometry, as Pohlke's Theorem is sometimes called, was given by H. A. Schwarz.* Subsequently numerous other proofs of the theorem and, in a few instances, of its generalization for an oblique system of coordinate-axes were given.†

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* *Crelle's Journal,* Vol. LXIII (1864), pp. 309–314, "Elementarer Beweis des Pohlkeschen Fundamentalsatzes der Axonometrie.

† von Deschwanden, who received his knowledge of the theorem from Steiner on one of the latter's visits to Zürich, gave a proof in the *Vierteljahrsschrift of the Maturforschende Gesellschaft in Zürich,* Vol. VI (1861), pp. 254–284, which, however, was not entirely satisfactory.

In the same volume, pp. 358–367, Kinkelin gave an analytic proof.

In Vol. XI, pp. 350–358, of the same publication, Reye, by means of projective geometry, generalized the theorem for oblique coordinates.

Among others who gave purely geometric demonstrations of the theorem, and constructive solutions of the problem involved may be mentioned:

Pelz, *Wiener Berichts,* Vol. LXXVI, II (1877), pp. 123–128.

Peschka, *Ibid.,* Vol. LXXVIII, II (1879), pp. 1043–1055.

Mandel, *Ibid.,* Vol. XCIV, II (1886), pp. 60–65.

Ruth, *Ibid.,* Vol. C, II (1891), pp. 1088–1092.

Schur, *Mathematische Annalen,* Vol. XXV (1885), pp. 596–597.

Schur, *Crelle's Journal,* Vol. CXVII (1896), pp. 474–475.

Küpper, *Mathematische Annalen,* Vol. XXXIII (1889), pp. 474–475.

Beck, *Crelle's Journal,* Vol. CVI (1890), pp. 121–124.

Schilling, *Zeitschrift für Mathematik und Physik,* Vol. XLVIII (1903), pp. 487–494.

Loria, *Vorlesungen über darstellende Geometrie,* Vol. I (1907), pp. 190–194,

Grossmann, *Darstellende Geometrie* (1915), pp. 26–29,

and various other well-known texts on descriptive geometry make use of the theorem in the discussion of axonometry.

I shall first investigate some of the properties of affine collineations in space, as far as they are related to the problem involved. Based upon these properties it will then not be difficult to prove Pohlke's and a number of similar theorems.

2. *Definition and General Properties of Affinity.*

Let OX, OY, OZ and $O'X'$, $O'Y'$, $O'Z'$ be two systems of coordinates, which, for the sake of definiteness, we assume as orthogonal; then the two spaces are defined as related by affinity when their coordinates are connected by the substitution

$$S \equiv \left\{ \begin{array}{l} x' = a_0 + a_1 x + a_2 y + a_3 z, \\ y' = b_0 + b_1 x + b_2 y + b_3 z, \\ z' = c_0 + c_1 x + c_2 y + c_3 z. \end{array} \right\} \tag{1}$$

The classification * of affinities depends upon the properties of the matrix

$$\left\| \begin{array}{llll} a_1-1 & a_2 & a_3 & a_0 \\ b_1 & b_2-1 & b_3 & b_0 \\ c_1 & c_2 & c_3-1 & c_0 \end{array} \right\|. \tag{2}$$

They form a projective twelve-parameter group and leave the plane at infinity invariant. Parallel planes and parallel lines are transformed into parallel planes and parallel lines. Of particular importance for our purpose is the case where the rank of matrix (2) is 1, so that the values of all its determinants of orders 3 and 2 vanish. The geometric meaning of this case is that the points of a certain plane s are left invariant, and that corresponding points lie on lines all parallel to a definite direction. Moreover, when P and P' are corresponding points and P_1 is the intersection of PP' with s, then $P'P_1 : PP_1 =$ constant. By a translation we can always make a_0, b_0, c_0 vanish, so that the origin $O \equiv O'$ becomes an invariant point. In this case the special affinity H, whose matrix is of rank 1, may always be written in the form

$$H \equiv \left\{ \begin{array}{l} x' = x + \lambda_1(x + py + qz), \\ y' = y + \lambda_2(x + py + qz), \\ z' = z + \lambda_3(x + py + qz), \end{array} \right\} \tag{3}$$

where $x + py + qz = 0$ is the plane, all of whose points are invariant. Corresponding points lie on parallel lines whose direction is determined by the constant ratios:

$$(x'-x)/(z'-z) = \lambda_1/\lambda_3, \quad (y'-y)/(z'-z) = \lambda_2/\lambda_3.$$

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* *Pascal's Repertorium*, Vol. II (2d ed.), pp. 100–101.

The line joining any two distinct corresponding points $P'(x', y', z')$, $P(x, y, z)$ cuts s in a point P_1 so that

$$P'P_1/PP_1 = 1+\lambda_1+\lambda_2 p+\lambda_3 q = \text{const.},$$

as stated above. This constant is also equal to the value of the determinant

$$\Delta = \begin{vmatrix} 1+\lambda_1 & \lambda_1 p & \lambda_1 q \\ \lambda_2 & 1+\lambda_2 p & \lambda_2 q \\ \lambda_3 & \lambda_3 p & 1+\lambda_3 q \end{vmatrix}$$

of the substitution H. When $\Delta=0$, then the homologous affinity H becomes a parallel projection on the plane s. The affinity is, in this case, singular.

Through every point $P(x, y, z)$ of an affinity S there is just one system of three mutually orthogonal planes which is transformed into such an orthogonal system through $P'(x', y', z')$. If these two systems are chosen as coordinate planes, S assumes the simple form

$$D \equiv x' = ax, \quad y' = by, \quad z' = cz, \tag{4}$$

which is called a dilatation. As the two coordinate systems (which we may assume as having both the same sense) may be brought to coincidence by a rotation R we have the well-known

THEOREM I. *Every affinity S may be considered as the product of a rotation R and a dilatation D, so that $S=RD$.**

3. *Homologous Affinity.*

We shall now consider in particular the affinity of type H. Such an affinity is also determined by two tetrahedrons whose corresponding points lie on four non-coplanar parallel lines. The planes of corresponding faces, and of corresponding planes, in general, meet in lines of a fixed plane s, the plane of homology. The parallel lines joining corresponding points pass through the same infinite point, called center of homology. We shall call H an homologous affinity.

The question is, whether it is possible to represent S in the form $S=R_1D_1H$, in which the substitution

$$D_1 \equiv x' = \rho x, \quad y' = \rho y, \quad z' = \rho z, \tag{5}$$

is a similitude.

For this purpose we first prove

THEOREM II. *There always exist homologous affinities by which any ellipsoid is transformed into a sphere, and conversely.*

Let $$x^2/a^2+y^2/b^2+z^2/c^2=1 \tag{6}$$

* *Pascal's Repertorium, loc. cit.* Koenigs, "Leçon de cinématique" (1897), pp. 394–405.

be any ellipsoid, and assume $a>b>c$. The two systems of circular sections are parallel to the diametral planes

$$z=\pm\frac{c}{a}\sqrt{\frac{a^2-b^2}{b^2-c^2}}\cdot x. \tag{7}$$

Consider the plane s, whose equation is obtained from (7) by choosing the — sign, and which cuts the ellipsoid in a circle. Through this as a great circle pass a sphere, whose equation will be

$$x^2+y^2+z^2=b^2. \tag{8}$$

It is easily shown that the ellipsoid (6) and the sphere (8) are inscribed in two right circular cylinders whose axes are in the xz-plane and have the slopes

$$m=\pm\sqrt{(b^2-c^2)/(a^2-b^2)}. \tag{9}$$

Denoting the coordinates of a point of the ellipsoid by x', y', z', and of a point on the sphere by x, y, z, and considering the cylinder obtained by taking the + sign in (9), it is found that the ellipsoid results from the sphere by the homologous affinity H, defined by

$$H^{-1}\equiv\left\{\begin{array}{l} x=x'-\dfrac{(ac+b^2)\sqrt{(a^2-b^2)(b^2-c^2)}}{ac\{a(b^2-c^2)+c(a^2-b^2)\}}\cdot\left\{c\sqrt{\dfrac{a^2-b^2}{b^2-c^2}}\cdot x'+az'\right\}, \\ y=y', \\ z=z'-\dfrac{(ac+b^2)(b^2-c^2)}{ac\{a(b^2-c^2)+c(a^2-b^2)\}}\cdot\left\{c\sqrt{\dfrac{a^2-b^2}{b^2-c^2}}\cdot x'+az'\right\}, \end{array}\right\} \tag{10}$$

with s as the invariant plane, and the direction in the xz-plane with the slope $m=+\sqrt{(b^2-c^2)/(a^2-b^2)}$ as that of the infinite center of homology. Two corresponding points P' and P of H are joined by a line cutting s in P_1, so that $P'P_1/PP_1=-\dfrac{ac}{b^2}=$ constant. The slope $m=-\sqrt{(b^2-c^2)/(a^2-b^2)}$ in (9) determines another homologous affinity with the same property, which is symmetrical with the first, with respect to the yz-plane. Now, an affinity transforms conjugate poles and polar planes and triplets of conjugate diameters of a quadric into corresponding poles and polar planes and triplets of conjugate diameters of the transformed quadric. Consequently, by the homologous affinity H^{-1} any three conjugate diameters of the ellipsoid (6) are transformed into three conjugate diameters of the sphere (8), which, as such, are orthogonal to each other. Likewise, any three conjugate radii OA', OB', OC' of the ellipsoid are transformed into three rectangular radii OA, OB, OC of the sphere. The lines AA', BB', CC' cut the sphere in three other points $A_1B_1C_1$, so that also OA_1, OB_1, OC_1 are orthogonal. The slope $-m$ determines two other

orthogonal trihedrals on the sphere, so that their extremities lie twice in sets on three parallel lines through A', B', C'.

But any three non-coplanar lines $A'A'_{-1}$, $B'B'_{-1}$, $C'C'_{-1}$, which bisect each other at O, as conjugate diameters uniquely determine an ellipsoid with O as a center. By Chasle's* or other† well-known methods the three rectangular-conjugate diametral planes, and the orthogonal-conjugate diameters, axes, may be constructed. Using these as coordinate axes, and denoting the semi-axes in the order of their magnitude by a, b, c, the equation of the ellipsoid may be written in the form (6). Then, by the method explained above we may construct the four orthogonal trihedrals on the corresponding affine sphere.

4. *A Certain Composition of Affinity.*

It is now possible to answer the question concerning the representation of a general affinity S in the form $S=R_1D_1H$.

According to Theorem I, let R be the rotation, and D the dilatation, so that $S=RD$ carries a point (x, y, z) into the point (x', y', z'). Around these points determine the corresponding orthogonal systems of coordinate axes, so that by S the sphere K, $x^2+y^2+z^2=1$, is transformed into the ellipsoid E, $x'^2/a^2+y'^2/b^2+z'^2/c^2=1$, with $a>b>c$. According to the method explained above construct the sphere K_2, so that E is obtained from K_2 by a homologous affinity H. Determine the orthogonal coordinate system G_2 through the center of K_2, corresponding to the orthogonal system through the center of E. The equation of K_2 with respect to G_2 is $x_2^2+y_2^2+z_2^2=b^2$. To G_2 apply the similitude

$$D_1^{-1}\equiv x_1=x_2/b, \quad y_1=y_2/b, \quad z_1=z_2/b. \tag{11}$$

Finally, by a definite rotation R_1^{-1} the system $G_1(x_1, y_1, z_1)$ is transformed into the original system $G(x, y, z)$. Conversely, by R_1 the sphere $x^2+y^2+z^2=1$ is transformed into the sphere K_1, $x_1^2+y_1^2+z_1^2=1$. By D_1, K_1 is transformed into the sphere K_2, $x_2^2+y_2^2+z_2^2=b^2$. By H, K_2 is transformed into the ellipsoid $x'^2/a^2+y'^2/b^2+z'^2/c^2=1$. With this the identity $S\equiv RD\equiv R_1D_1H$ is proved for non-singular affinities.

It is also true for a singular affinity S_s, in which all points (x, y, z) are transformed into points (x', y', z') which lie in a plane. If we choose this as the plane s, the dilatation D_s will have the form

$$D_s\equiv x'=ax, \quad y'=by, \quad z'=0z. \tag{12}$$

* Beck, *loc. cit.*

† Fiedler, *Darstellende Geometrie*, Vol. II (1885), pp. 329–330.

The homologous affinity H becomes a parallel projection upon a plane s, defined by

$$H_s \equiv x' = x - \frac{\sqrt{a^2-b^2}}{b} \cdot z, \quad y'=y, \quad z'=0, \tag{13}$$

as is easily found from (10), by solving for x', y', z', and letting lim $(c)=0$. The ellipsoid becomes an infinitely thin disk E_s in the plane s, whose contour has the equation $x'^2/a^2+y'^2/b^2=1$, and we find again $S_s=R_1 \cdot D_1 \cdot H_s$. Hence

THEOREM III. *Every general affinity in space is the product of a rotation, a similitude, and an homologous affinity. In case of a singular affinity, in which all points are transformed into points of a plane, the homologous affinity becomes a parallel projection upon this plane.*

5. *Pohlke's Theorem and its Generalization.*

Chasle's construction of the axes of an ellipsoid still holds when the three conjugate diameters $A'A'_{-1}$, $B'B'_{-1}$, $C'C'_{-1}$ are coplanar. As before, the three diametral planes cut the degenerate ellipsoid E_s in three ellipses $(A'B')$, $(B'C')$, $(C'A')$, which are inscribed in three parallelograms as shown in Fig. 1.

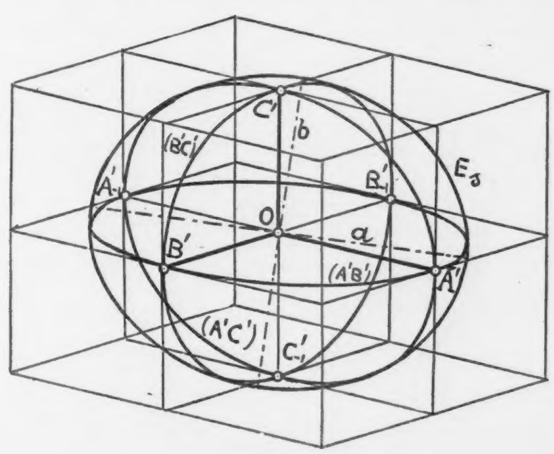

FIG. 1.

These ellipses are inscribed in an ellipse E_s in s, whose half-axes we denote by a and b $(a>b)$, and whose lines we choose as x'- and y'-axes, and the line through the center of E_s, perpendicular to s, as the z'-axis. Then the equation of E_s is precisely that given above as $x'^2/a^2+y'^2/b^2=1$. The sphere K with the radius b, concentric with E_s, is now projected into E_s by a parallel projection H_s as defined by (13). Conversely, by the same formulas for H_s, and geometrically, it is easily verified, that to the ellipses $(A'B')$, $(B'C')$, $(C'A')$,

and their circumscribed parallelograms, correspond on K three great circles, whose planes are mutually perpendicular, and their circumscribed squares. To the complete rhombohedral lattice-work, circumscribed and inscribed to the ellipsoid as before, corresponds a cubical lattice-work connected in the same manner to the sphere, so that the rhombohedral is the parallel-projection (H_s)

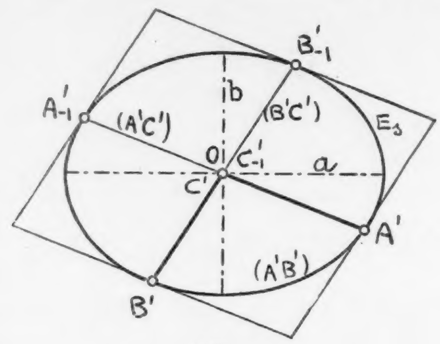

FIG. 2.

of the cubical lattice work. In this manner to the three distinct conjugate coplanar radii OA', OB', OC', correspond the three equal orthogonal radii OA, OB, OC. There are, in general, again four such sets of orthogonal radii.

The proposition is still true when one of the three coplanar radii, say OC', vanishes. The ellipse $(A'B')$ has $A'A'_{-1}$ and $B'B'_{-1}$ as conjugate diameters, while the ellipses $(B'C')$ and $(C'A')$ coincide with the segments $B'B'_{-1}$ and $A'A'_{-1}$, Fig. 2. The contour ellipse E_s coincides with the ellipse $(A'B')$. The

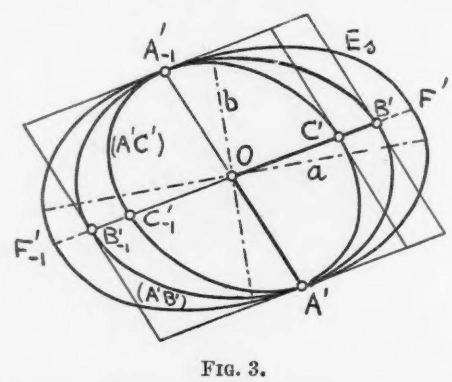

FIG. 3.

homologous sphere may be constructed precisely as before, so that to the ellipses $(A'B')$, ($B'C'=$ degenerate), ($C'A'=$ degenerate) correspond again three orthogonal great circles on the sphere K, and to the coplanar conjugate semi-diameters OA', OB', $OC'=0$, three orthogonal radiï of K.

When one of the angles, say the one between $B'B'_{-1}$ and $C'C'_{-1}$ is zero, Fig. 3, then the ellipses $(A'B')$, $(A'C')$ may be constructed as in the general

case. The ellipse $(B'C')$ degenerates into a straight line segment $F'F'_{-1}$, so that $|OF'| = |OF'_{-1}| = \sqrt{\overline{OB'}^2 + \overline{OC'}^2}$. In this case E_s is the ellipse which has $A'A'_{-1}$ and $F'F'_{-1}$ as conjugate diameters, and which with the ellipses $(A'B')$, $(B'C')$, $(C'A')$, and the corresponding circumscribed parallelograms and conjugate diameters may be considered as the parallel projection H_s of a sphere K with three orthogonal great circles and the attached cubic lattice-work. In every case there exist equiaxial-orthogonal trihedrals ($OA = OB = OC$; $\sphericalangle AOB = \sphericalangle BOC = \sphericalangle COA = 90°$) of which OA', OB', OC', whether coplanar or not, form a parallel projection. Hence, we may state Pohlke's Theorem in a generalized form.

THEOREM IV. *The vertex and the extremities of any three concurrent, coplanar or non-coplanar, straight line segments in space always lie in a definite order on four parallel lines through the vertex and the extremities of an equiaxial-orthogonal trihedral. In general, there are two distinct sets of four parallel lines each, and four sets of orthogonal trihedrals with this property. Not more than one segment, and not more than one angle between the segments of the given trihedral may vanish.*

As a general affinity S depends on twelve independent parameters, it is always possible to determine uniquely an affinity S in which any two proper tetrahedrons T' and T correspond to each other in a definite order. For example, $P'_1P'_2P'_3P'_4$ to $P_1P_2P_3P_4$. But we have proved that $S = R_1D_1H$, so that T' results from T by a rotation, followed by a similitude, and finally by an homologous affinity. When $P'_1P'_2P'_3P'_4$ are coplanar, then the substitution H becomes a parallel projection H_s, and S is a singular affinity, for which the determinant of the substitution vanishes. The result may be stated as

THEOREM V.* *If any two proper tetrahedrons $P'_1P'_2P'_3P'_4$ and $P_1P_2P_3P_4$ are given, it is always possible to determine a tetrahedron $P''_1P''_2P''_3P''_4$ similar (eventually congruent) to $P_1P_2P_3P_4$, so that the lines joining P'_1 and P''_1, P'_2 and P''_2, P'_3 and P''_3, P'_4 and P''_4 are parallel. This is still true when the points $P'_1P'_2P'_3P'_4$ form a proper plain quadrangle, or also when the segments $P'_4P'_1$, $P'_4P'_2$, $P'_4P'_3$ and the angles formed by them are subject to the necessary and sufficient conditions of Pohlke's Theorem.*

This theorem clearly contains Pohlke's and Reye's theorems as special cases.

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*The first part of this theorem concerning two proper tetrahedrons has also been proved by Hurwitz in a recent communication to the Swiss Mathematical Society, an abstract of which in *L'Enseignement Mathématique* reached the author several months after this paper was sent to the AMERICAN JOURNAL OF MATHEMATICS.

6. *Related Theorems.*

From the connection between the rhombohedral and cubical lattice-works discussed above, we deduce without difficulty

THEOREM VI. *A plain hexagon with three pairs of parallel, opposite sides, with the sides of each pair equal, may always be considered as the contour of a parallel projection of a cube. The net of six parallelograms constructed with each two adjacent sides of the hexagon as a pair of adjacent sides of a parallelogram, is the projection of the edges of the cube.*

Completing the rhombohedral lattice-works, determined by P_4P_1, P_4P_2, P_4P_3 and $P'_4P'_1$, $P'_4P'_2$, $P'_4P'_3$ as clinographic semi-axes of the rhombohedrons, and inscribing ellipsoids into these, with the clinographic axes in each case as triplets of conjugate diameters, we find

THEOREM VII. *If any two parallelopipeds (rhombohedrons) π' and π are given, it is always possible to find a parallelopiped π'' similar (eventually congruent) to π, so that corresponding vertices of π' and π'' lie on eight parallel lines (eventually counting multiplicities properly).*

In a similar manner we have

THEOREM VIII. *If any two ellipsoids E' and E are given, it is always possible to find an ellipsoid E'' similar (eventually congruent) to E, so that E' and E'' are inscribed to the same elliptic (circular) cylinder.*

Likewise as a special case of the foregoing,

THEOREM IX. *The contour of a parallel projection of any given ellipsoid upon a plane may be similar to any given ellipse.*

Finally,

THEOREM X. *It is always possible to circumscribe two (may be coincident) right circular cylinders to any ellipsoid.*

Arithmetical Theory of Certain Hurwitzian Continued Fractions.

By D. N. Lehmer.

Introduction.

The following investigation is the outcome of the discovery, made some three years ago, of the curious fact that the denominator of the convergent of order $3n$ in the regular continued fraction which represents the base of Naperian logarithms is divisible by n. It was later found that the same is true of the denominators of the convergents of order $3n-2$ and $3n-6$, and of the numerator of the convergent of order $3n-3$. Further, the convergents were found to recur with a period of $3n$ terms, or of $6n$ terms according as n is even or odd.

These theorems, discovered empirically, turned out to be remarkably intractable, and, although, a year ago, a method was discovered of establishing them, it would not apply to other continued fractions of the same general type for which the same or similar theorems seemed to hold.

The discovery of the regular continued fraction

$$e=(2, 1, 2, 1, 1, 4, 1, 1, 6, 1, 1, 8, 1, 1, 10, 1, 1, \ldots\ldots),$$

for the base of Naperian logarithms is credited to Roger Cotes,* but to Euler † is due the rediscovery of it, and the general proof of the law of the successive partial quotients by means of the solution of Riccati's equation. Euler also found other remarkable continued fractions such as

$$e=(1, 1, 1, 1, 5, 1, 1, 9, 1, 1, 14, 1, 1, \ldots\ldots),$$

$$(e^{\frac{1}{s}}+1)/(e^{\frac{1}{s}}-1)=(2s, 6s, 10s, 14s, \ldots\ldots),$$

$$e^{\frac{1}{s}}=(1, s-1, 1, 1, 3s-1, 1, 1, 5s-1, 1, 1, \ldots\ldots).$$

Hurwitz ‡ has studied a very general type of continued fraction, to which the above fractions all belong. He makes use of the notation

$$(q_1, q_2, \ldots\ldots, q_r, \overline{f_1(m), f_2(m), f_3(m), \ldots\ldots, f_k(m)}), \quad (m=0, 1, 2, 3, \ldots\ldots),$$

for the continued fraction whose partial quotients are

$$q_1, q_2, \ldots\ldots, q_r, f_1(0), f_2(0), \ldots\ldots, f_k(0), f_1(1), f_2(1), \ldots\ldots, f_k(1), f_1(2), f_2(2), \ldots\ldots,$$

* Cotes, "Logometria," *Phil. Trans.*, London (1714), Vol. XXIX, p. 5.

† Euler, *Comm. Acad. Petrop.* (1737), p. 121, edition of 1744.

‡ Hurwitz, *Vierteljahrschrift der Naturforschenden Gesellschaft in Zurich* (1896), Vol. XLI, p. 34.

where the q's are rational, and with the possible exception of q_1, all positive. The functions f are rational, integral functions whose degrees may some or all be zero. If, however, the degrees are all zero, the fraction becomes an ordinary periodic continued fraction. If the highest degree found among any of the functions is m, the fraction is said to be of the m-th order. An ordinary periodic continued fraction is thus a Hurwitzian fraction of order zero. Written in this notation the above fractions of Euler read:

$$e=(2, \overline{1, 2m+2, 1}), \qquad (m=0, 1, 2, 3, \ldots.),$$
$$(e^{\frac{1}{s}}+1)/(e^{\frac{1}{s}}-1)=[\overline{(4m+2)s}], \qquad (m=0, 1, 2, 3, \ldots.),$$
$$e^{\frac{1}{s}}=[\overline{1, (2m+1)s, 1)}], \qquad (m=0, 1, 2, 3, \ldots.),$$

Hurwitz has shown that if the irrational numbers ξ and η are connected by the equation $\xi=(\alpha\eta+\beta)/(\gamma\eta+\delta)$, where α, β, γ, δ are integers such that $\alpha\delta-\beta\gamma$ is not zero, then if the regular continued fraction for η is of the Hurwitzian type, so also is that for ξ, and the functions f appearing in each expansion are of the same degrees, with the possible exception of those of zero degree which may appear in one and not in the other.

In this paper we shall deal with Hurwitzian fractions in which the functions f are all of degree zero except one, and that one is of the first degree. We shall write the fraction in the form:

$$(q_1, q_2, \ldots., q_r, \overline{a_1, a_2, a_3, \ldots., a_{k-1}, bm+c}), \qquad (m=1, 2, 3, \ldots.),$$

and taking first the case where the q's are all absent, we show that, with certain interesting exceptions, it is true for all such fractions that the numerator of the convergent of order $2nk-1$, and the denominator of the convergent of order $2nk$ are divisible by n, while the numerator of the convergent of order $2nk$, and the denominator of the convergent of order $2nk-1$ are congruent modulo n to $(-1)^{nk-1}$, so that the series of convergents repeat themselves, modulo n after $4nk$ terms, or after $2nk$ terms according as nk is even or odd.

Exceptions to this rule occur when b is congruent to zero, modulo n, or when the numerator or denominator of the convergent of order $k-1$ is congruent to zero, modulo n. The period of the convergents is not so simply stated for these cases.

The laws for the period of the fraction when the q's are actually present are easily obtainable.

In this paper we are not considering questions of convergence or divergence of continued fractions. Certain of the fractions involved are closely related to those called "semiregular" whose convergence has been studied by

Tietze,* and the rules for determining convergence or divergence of semi-regular continued fractions may be modified to apply to them. We are concerned here with the successive values of the numerators and denominators of the convergents, and not with the existence or non-existence of a limiting value to those convergents. The theorems obtained have to do with numbers which satisfy certain difference equations of the second order, and are thus, as Professor Birkhoff has remarked, extensions of Wilson's and Fermat's theorems, which have to do with numbers which satisfy the difference equations $u_{n+1}=nu_n$ and $u_{n+1}=au_n$ respectively.

I.

Let A_m/B_m be the m-th convergent of the continued fraction

$$\overline{(a_1,\, a_2,\, a_3,\, \ldots,\, a_{k-1},\, \mu b)}, \qquad (\mu=1, 2, 3, \ldots), \tag{1}$$

where all the partial quotients are positive or negative integers or zero. Let also A'_m/B'_m be the m-th convergent of the fraction

$$\overline{(a_{k-1},\, a_{k-2},\, \ldots,\, a_2,\, a_1,\, -b\mu-2M)}, \qquad (\mu=1, 2, \ldots), \tag{2}$$

where

$$M=(A_{k-2}+B_{k-1})/A_{k-1}, \tag{3}$$

and we will suppose that A_{k-1} is not zero. We will show by complete induction that the following equations hold:

$$A_{rk-1}=A'_{rk-1}(-1)^{r-1}, \tag{4}$$

$$A_{rk}=(A'_{rk}+MA'_{rk-1})(-1)^r. \tag{5}$$

By using equation (4) we may interchange A and A' in (5).

To start the proof it is necessary to show that the formulae hold for $r=1$. The fractions $(a_1, a_2, a_3, \ldots, a_{k-1})$ and $(a_{k-1}, \ldots, a_2, a_1)$ are inverse. (See Perron, "Die Lehre von den Kettenbrüchen," p. 32). We have, therefore, $A'_{k-1}/A'_{k-2}=(a_1, a_2, \ldots, a_{k-1})=A_{k-1}/B_{k-1}$, and since the fractions are in their lowest terms,

$$A_{k-1}=A'_{k-1}, \tag{6}$$

$$B_{k-1}=A'_{k-2}; \tag{7}$$

and similarly,

$$A_{k-2}=B'_{k-1}, \tag{8}$$

$$B_{k-2}=B'_{k-2}. \tag{9}$$

Equation (4) therefore holds when $r=1$, by (6). Also, from the definition of M we have, using (6) again, $MA'_{k-1}=A_{k-2}+B_{k-1}$, so that when $r=1$, (5) becomes

$$A_k=-(A'_k+A_{k-2}+B_{k-1}). \tag{10}$$

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* Tietze, *Math. Ann.* (1911), Vol. LXX.

But from the definitions of the continued fractions themselves we have

$$A_k = bA_{k-1} + A_{k-2}, \tag{11}$$

and

$$A'_k = -(b+2M)A'_{k-1} + A'_{k-2}; \tag{12}$$

or, using (6) and (7),

$$A'_k = -(bA_{k-1} + 2A_{k-2} + B_{k-1}). \tag{13}$$

Substituting (11) and (13) in (10) it is found that equation (5) is true when $r=1$.

We now show that if equations (4) and (5) are assumed to hold for all values of r up to and including $r=n$, they must also hold for $r=n+1$.

By the fundamental formula of continued fractions (see Perron, *loc. cit.*, p. 14),

$$A_{(n+1)k-1} = A_{k-1}A_{nk} + B_{k-1}A_{nk-1}, \tag{14}$$

and assuming formulae (4) and (5) for $r=n$ this gives

$$A_{(n+1)k-1} = (-1)^n A_{k-1}(A'_{nk} + MA'_{nk-1}) + (-1)^{n-1}A'_{nk-1}B_{k-1},$$

and, recalling the definition of M, this becomes

$$A_{(n+1)k-1} = (-1)^n(A_{k-1}A'_{nk} + A_{k-2}A'_{nk-1}),$$

and using (6) and (8) this is $A_{(n+1)k-1} = (-1)^n(A'_{k-1}A'_{nk} + B'_{k-1}A'_{nk-1})$, which, by the fundamental formula (14), gives

$$A_{(n+1)k-1} = (-1)^n A'_{(n+1)k-1}, \tag{15}$$

which is formula (4) for $r=n+1$.

Suppose now that formula (5) holds for $r=n$, so that

$$A_{nk} = (-1)^n(A'_{nk} + MA'_{nk-1}).$$

Multiply both sides of this equation by A_{k-2}, and add $B_{k-2}A_{nk-1}$ to the left-hand side, and the equal expression, $(-1)^{n-1}B_{k-2}A'_{nk-1}$ to the right. We thus obtain

$$A_{k-2}A_{nk} + B_{k-2}A_{nk-1} = (-1)^n(A_{k-2}A'_{nk} + MA_{k-2}A'_{nk-1} - B_{k-2}A'_{nk-1}).$$

Using (6), (7), (8) and (9), we can throw this into the form

$$A_{k-2}A_{nk} + B_{k-2}A_{nk-1} = (-1)^n[M(A'_{k-1}A'_{nk} + B'_{k-1}A'_{nk-1}) - (A'_{k-2}A'_{nk} + B'_{k-2}A'_{nk-1})].$$

But by the fundamental formulae of continued fractions (Perron, *loc. cit.*, p. 14), this last equation may be written $A_{(n+1)k-2} = (-1)^n(MA_{(n+1)k-1} - A_{(n+1)k-2})$. To the left side of this equation add the term $(n+1)bA_{(n+1)k-1}$, and to the right side add the term $(-1)^n(n+1)bA'_{(n+1)k-1}$, which by equation (15) is legitimate, and the result is

$$(n+1)bA_{(n+1)k-1} + A_{(n+1)k-2}$$
$$= (-1)^{n+1}[-(n+1)bA'_{(n+1)k-1} - MA'_{(n+1)k-1} + A'_{(n+1)k-2}]. \tag{16}$$

But from the recurrent relation connecting the numerators in the continued fraction we have $A_{(n+1)k}=(n+1)A_{(n+1)k-2}+A_{(n+1)k-2}$, and

$$A'_{(n+1)k}=-[(n+1)b+2M]A'_{(n+1)k-1}+A'_{(n+1)k-2}.$$

Putting these values in (16) we get $A_{(n+1)k}=(-1)^{n+1}(A'_{(n+1)k}+MA'_{(n+1)k-1})$, which is equation (5) when $r=n+1$. If then (4) and (5) are true for $r=n$, they must be true for $r=n+1$. But they have been shown to hold for $r=1$, therefore they hold in general.

II.

Consider now the continued fraction

$$(x, a_{k-1}, a_{k-2}, \ldots, a_2, a_1, y, \overline{a_{k-1}, a_{k-2}, \ldots, a_2, a_1, -b\mu-2M}), \qquad (\mu=1, 2, \ldots), \quad (17)$$

where
$$x=(B_{k-1}-A_{k-2})/A_{k-1}, \quad (18)$$
and
$$y=-(2B^2_{k-1}+A_{k-1}B_k+A_{k-1}B_{k-2})/A_{k-1}B_{k-1}. \quad (19)$$

We assume that A_{k-1} and B_{k-1} are different from zero. We denote the $(m+1)$st convergent of this fraction by A''_m/B''_m, and show by complete induction that the following equations hold:

$$B_{rk-1}=A''_{rk-1}(-1)^{r-1}, \quad (20)$$

$$B_{rk}=(A''_{rk}+MA''_{rk-1})(-1)^r. \quad (21)$$

Equation (21) may be written, using (20),

$$A''_{rk}=(B_{rk}+MB_{rk-1})(-1)^r. \quad (22)$$

We first show that the formulae hold when $r=1$. We have $A''_{k-1}/B''_{k-1}-x=(0, a_{k-1}, a_{k-2}, \ldots, a_1)$, or $B''_{k-1}/(A''_{k-1}-xB''_{k-1})=(a_{k-1}, \ldots, a_1)=A'_{k-1}/B'_{k-1}=A_{k-1}/A_{k-2}$, and since all fractions are in their lowest terms,

$$B''_{k-1}=A_{k-1}, \quad (23)$$

$$A''_{k-1}-xB''_{k-1}=A_{k-2}. \quad (24)$$

From these two equations with (18) we get at once

$$A''_{k-1}=B_{k-1}, \quad (25)$$

which is formula (20) when $r=1$.

In the same way, using the relations $B''_{k-2}/(A''_{k-2}-xB''_{k-2})=(a_{k-1}, \ldots, a_2)=A'_{k-2}/B'_{k-2}=B_{k-1}/B_{k-2}$, we get

$$B''_{k-1}=B_{k-1}, \quad (26)$$

$$A''_{k-2}=xB_{k-1}+B_{k-2}, \quad (27)$$

whence, using the defining equation for x again, and remembering from the general theory of continued fractions that

$$A_{k-1}B_{k-2}-B_{k-1}A_{k-2}=(-1)^{k-1}, \quad (28)$$

Equation (27) reduces to

$$A''_{k-2}=[B^2_{k-1}+(-1)^{k-1}]/A_{k-1}. \tag{29}$$

Now $A''_k=yA''_{k-1}+A''_{k-2}$, and using the defining equation for y together with (25) and (27) we get from this,

$$A''_k=-[B^2_{k-1}+A_{k-1}(B_k+B_{k-2})-(-1)^{k-1}]/A_{k-1}. \tag{30}$$

Recall now the definition of M and we get

$$A''_k+MA''_{k-1}=-[B^2_{k-1}+A_{k-1}(B_k+B_{k-2})-A_{k-2}B_{k-1}-(-1)^{k-1}-B^2_{k-1}]/A_{k-1},$$

which reduces again, using (28), to

$$A''_k+MA''_{k-1}=-B_k, \tag{31}$$

which agrees with formula (21) when $r=1$.

We now show that if formulae (20) and (21) are true for all values of r up to and including $r=n$ they must hold for $r=n+1$. We have, using again the formulae given in Perron, page 14, $A_{(n+1)k-1}=A_{k-1}A_{nk}+B_{k-1}A_{nk-1}$, and using formulae (20) and (21), which are assumed to hold for $r=n$, this may be written: $A''_{(n+1)k-1}=(-1)^nA_{k-1}(B_{nk}+MB_{nk-1})+(-1)^{n-1}A_{k-2}B_{nk-1}$. Putting in the value of M we get from this,

$$A''_{(n+1)k-1}=(-1)^n(A_{k-1}B_{nk}+B_{k-1}B_{nk-1})=(-1)^nB_{(n+1)k-1},$$

which is formula (20) when $r=n+1$.

Starting now with the equation which comes from the way the continued fraction is defined,

$$B_{(n+1)k}=(n+1)bB_{(n+1)k-1}+B_{(n+1)k-2}, \tag{32}$$

we write it, using the fundamental recursion formulae (Perron, p. 14),

$$B_{(n+1)k}=(n+1)bB_{(n+1)k-1}+A_{k-2}B_{nk}+B_{k-2}B_{nk-1}.$$

This may again be written (since $B_{(n+1)k-1}=A_{k-1}B_{nk}+B_{k-1}B_{nk-1}$),

$$B_{(n+1)k}=[(n+1)b+M]B_{(n+1)k-1}-B_{k-1}(B_{nk}+MB_{nk-1})+B_{k-2}B_{nk-1}, \tag{33}$$

but for $r=n$ we have, by hypothesis, using (22) and (20),

$$B_{nk}+MB_{nk-1}=A''_{nk}(-1)^n,\quad B_{nk-1}=(-1)^{n-1}A''_{nk-1}.$$

Also, by (7) and (9), $B_{k-1}=A'_{k-2}$, $B_{k-2}=B'_{k-2}$. Putting these values in (33) we get

$$B_{(n+1)k}=[(n+1)b+M]B_{(n+1)k-1}+(-1)^{n+1}(A'_{k-2}A''_{nk}+B'_{k-2}A''_{nk-1}), \tag{34}$$

but again, from the construction of the fraction $A''_{(n+1)k-2}=A'_{k-1}A''_{nk}+B'_{k-2}A''_{nk-1}$, and (34) becomes

$$B_{(n+1)k}=[(n+1)b+M]B_{(n+1)k-1}+(-1)^{n+1}A''_{(n+1)k-2}. \tag{35}$$

Now we have already shown that formula (20) holds for $r=n+1$ so that we can write $B_{(n+1)k-1}=(-1)^nA''_{(n+1)k-1}$ in (35), and get

$$B_{(n+1)k}=(-1)^{n+1}[((n+1)b+M)A''_{(n+1)k-1}+A''_{(n+1)k-2}]. \tag{36}$$

But again, from the succession of partial quotients of the continued fraction, $A''_{(n+1)k}=-[(n+1)b+2M]A''_{(n+1)k-1}+A''_{(n+1)k-2}$, $(n=1,2,3,\ldots.)$. This in (36) gives $B_{(n+1)k}=(-1)^{n+1}(A''_{(n+1)k}+MA''_{(n+1)k-1})$, which is formula (21) when $r=n+1$. These formulae therefore hold in all cases.

From the definition of M it would seem that k must not be less than 3. With the usual conventions, however, that $A_0=1$, $B_0=0$, $A_{-1}=0$ and $B_{-1}=1$, the theorems derived above will apply when $k=1$ and $k=2$.

III.

We consider now the continued fractions (1), (2) and (17) with respect to any modulus n, and we assume that n is prime to $2b$ and to A_{k-1} and to B_{k-1}. The cases where these restrictions are not applied will be considered later. It is then possible to find two values of μ, one odd and the other even, both less than $2n$, which will satisfy the congruence

$$b\mu+2M\equiv 0 \pmod{n}. \tag{37}$$

Such a solution will furnish a zero partial quotient in the continued fraction (2) of rank μk, and the partial quotient of the same rank in fraction (1) will have the value $-2M$. Moreover, it is seen that the partial quotients of (2) read backward from this partial quotient exactly as the partial quotients of (1) read forward, so that, taken modulo n, the two fractions are inverse as far as this partial quotient. We consider first the even solution of (37), which we denote by $2m$. By the properties of inverse fractions we have (see equations (6), (7), (8) and (9)) $A_{2mk-1}\equiv A'_{2mk-1} \pmod{n}$. But by equation (4), $A_{2mk-1}=-A'_{2mk-1}$, so that

$$A_{2mk-1}\equiv 0 \pmod{n}. \tag{38}$$

IV.

The partial quotient of rank $2m$ is, as we noted above, congruent to $-2M$, the two preceding ones being a_{k-1} and a_{k-2}. The recursion formulae for a continued fraction give

$$A_{2mk}=-2MA_{2mk-1}+A_{2mk-2}, \tag{39}$$

$$A_{2mk-1}=a_{k-1}A_{2mk-2}+A_{2mk-3}, \tag{40}$$

$$A_{2mk-2}=a_{k-2}A_{2mk-3}+A_{2mk-4}. \tag{41}$$

From these we derive, using (38) the congruences,

$$\left.\begin{aligned} &A_{2mk-2}\equiv A_{2mk}\equiv A'_0A_{2mk}, \\ &A_{2mk-3}\equiv -a_{k-1}A_{2mk}\equiv -A'_1A_{2mk}, \\ &A_{2mk-4}\equiv (a_{k-1}a_{k-2}+1)A_{2mk}\equiv A'_2A_{2mk}, \\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots, \end{aligned}\right\} \pmod{n}.$$

We infer the general law

$$A_{2mk-r} \equiv (-1)^r A'_{r-2} A_{2mk} \pmod{n}. \tag{42}$$

The proof is made by complete induction. We see that the law holds for $r=1$. Suppose it true for all values of r up to and including $r=t$. Then since the continued fractions (1) and (2) are inverse modulo n as far as the partial quotient of order $2mk$ the partial quotient opposite A_{2mk-t} is the same, modulo n, as that opposite A'_t. But we have $A_{2mk-(t+1)} = A_{2mk-(t-1)} - q_{t-1} A_{2mk-t}$, where q_{t-1} is the partial quotient opposite $A_{2mk-(t-1)}$. By means of (42), which by hypothesis holds for $r=t$ this last equation may be written:

$$\begin{aligned} A_{2mk-(t+1)} &\equiv A_{2mk}[A'_{t-3}(-1)^{t-1} - q_{t-1}A'_{t-2}(-1)^t] \\ &\equiv (-1)^{t-1} A_{2mk}(A'_{t-3} + q_{t-1}A'_{t-2}) \equiv (-1)^{t-1} A_{2mk} A'_{t-1}, \end{aligned}$$

which is formula (42) for $r=t+1$. The formula then holds for all values of r. It may also be written

$$A_{2mk-(r+2)} \equiv (-1)^r A_{2mk} A'_r \pmod{n}. \tag{43}$$

Returning with this result to equation (4) we obtain the congruence

$$\begin{aligned} A_{2mk-(kr+1)} &\equiv (-1)^{kr+1} A_{2mk} (-1)^{r-1} A_{kr-1} \pmod{n} \\ &\equiv (-1)^{r(k+1)} A_{2mk} A_{kr-1} \pmod{n}. \end{aligned} \tag{44}$$

Put $r=m$ in this congruence and get $A_{mk-1} \equiv (-1)^{m(k+1)} A_{2mk} A_{mk-1} \pmod{n}$, or

$$A_{mk-1}[A_{2mk} - (-1)^{m(k+1)}] \equiv 0 \pmod{n}. \tag{45}$$

In the same way, starting with equation (5), we get the congruence

$$(-1)^{r(k-1)} A_{2mk} A_{rk} \equiv A_{2mk-rk-2} - M A_{2mk-rk-1} \pmod{n},$$

and putting $r=m$ in this we get $(-1)^{m(k-1)} A_{2mk} A_{mk} \equiv A_{mk-2} - M A_{mk-1}$. But $A_{mk} = mbA_{mk-1} + A_{mk-2}$, whence $(-1)^{m(k-1)} A_{2mk} A_{mk} \equiv A_{mk} - (mb+M) A_{mk-1}$, which we may write in the form

$$A_{mk}[(-1)^{m(k-1)} - A_{2mk}] \equiv (mb+M) A_{mk-1} \pmod{n}. \tag{46}$$

Now by (45) either A_{mk-1} is congruent to zero, or else $A_{2mk} - (-1)^{m(k-1)}$ is congruent to zero, or perhaps both factors are. But if A_{mk-1} is congruent to zero then A_{mk} is not, since, by the equation $A_{mk}B_{mk-1} - A_{mk-1}B_{mk} = (-1)^{mk}$, A_{mk} and A_{mk-1} can have no common factor. Therefore by (46), if A_{mk-1} is congruent to zero, so is $A_{2mk} - (-1)^{m(k-1)}$ also. If, on the other hand, A_{mk-1} is not congruent to zero, then by (45) $A_{2mk} - (-1)^{m(k-1)}$ must be. Therefore we have in all cases,

$$A_{2mk} \equiv (-1)^{m(k-1)} \pmod{n}. \tag{47}$$

V.

Returning to (42) with this last result, we get

$$A_{2mk-r} \equiv (-1)^{r+m(k+1)} A'_{r-2} \pmod{n}. \tag{48}$$

VI.

From the equation $A_{2mk}B_{2mk-1}-A_{2mk-1}B_{2mk}=1$ we have, using (38) and (47),

$$B_{2mk-1}\equiv(-1)^{m(k-1)} \pmod{n}. \tag{49}$$

VII.

Combining (47) with (44) we obtain

$$A_{2mk-rk+1}\equiv(-1)^{(k+1)(m+r)}A_{rk-1} \pmod{n}. \tag{50}$$

This formula shows that apart from sign the values of A_{rk-1} read backward and forward the same, from $r=0$ to $r=2mk-1$. When k is odd the signs are all the same, while if k is even the signs alternate. It is easily shown that the same theorems apply to the continued fraction (2). Corresponding theorems hold for the denominators B_{rk-1}, but to establish them we must consider continued fraction (17).

VIII.

It will be observed that after the partial quotient y in (17), the succession of partial quotients are the same as in (2). Let us call P_r/Q_r the r-th convergent to the fraction

$$\overline{[a_{k-1}, a_{k-2}, \ldots, a_2, a_1, -(b\mu+2M)]}, \quad (\mu=2, 3, \ldots). \tag{51}$$

We have then,

$$A''_r=A''_kP_r+A''_{k-1}Q_r, \tag{52}$$

$$B''_r=B''_kP_r+B''_{k-1}Q_r, \tag{53}$$

$$A'_r=A'_kP_r+A'_{k-1}Q_r, \tag{54}$$

$$B'_r=B'_kP_r+B'_{k-1}Q_r. \tag{55}$$

Solving (52) and (53) for P_r and Q_r we get

$$P_r=(-1)^k(B''_{k-1}A''_r-A''_{k-1}B''_r), \tag{56}$$

$$Q_r=(-1)^k(-B''_kA''_r+A''_kB''_r). \tag{57}$$

Similarly, from (54) and (55),

$$P_r=(-1)^k(B'_{k-1}A'_r-A'_{k-1}B'_r), \tag{58}$$

$$Q_r=(-1)^k(-B'_kA'_r+A'_kB'_r). \tag{59}$$

From (56) and (58) we get

$$B''_{k-1}A''_r-A''_{k-1}B''_r=B'_{k-1}A'_r-A'_{k-1}B'_r, \tag{60}$$

and from (57) and (59) we get

$$-B''_kA''_r+A''_kB''_r=-B'_kA'_r+A'_kB'_r. \tag{61}$$

Eliminate B''_r from (60) and (61) and we get

$$B'_r(A''_{k-1}A'_k-A'_{k-1}A''_k)-A'_r(A''_{k-1}B'_k-A''_kB'_{k-1})=(-1)^kA''_r. \tag{62}$$

We proceed to find the values of the quantities in the parentheses. We know from (6) that $A'_{k-1}=A_{k-1}$. Also from (30),

$$A''_k=-[B^2_{k-1}+A_{k-1}(B_k+B_{k-2})-(-1)^{k-1}]/A_{k-1};$$

while from (13) $A'_k=-(bA_{k-1}+2A_{k-2}+B_{k-1})$, and from (25) $A''_{k-1}=B_{k-1}$. Putting these values in the coefficient of B'_r in (62), replacing B_k by its value $bB_{k-1}+B_{k-2}$, and remembering that $A_{k-1}B_{k-2}-A_{k-2}B_{k-1}$ is equal to $(-1)^{k-1}$, we get easily $A''_{k-1}A'_k-A''_kA'_{k-1}=(-1)^{k-1}$. The coefficient of A'_r also reduces. For we have $B'_k=-(b+M)B'_{k-2}+B'_{k-2}$; or, using (8) and (9),

$$B'_k=-(b+M)A_{k-2}+B_{k-2}.$$

Using the same reductions as before, the coefficient of A'_r may be made to take the form $(-1)^{k-1}M$, so that (62) reduces to

$$A''_r=MA'_r-B'_r. \tag{63}$$

If now in this last equation we put $rk-1$ in place of r, and make use of (20) and (50) this may be written:

$$(-1)^rB'_{rk-1}\equiv B_{rk-1}-MA_{rk-1} \pmod n. \tag{64}$$

Again, put $r=2km$ in (63), and note that $A'_{2mk}\equiv(-1)^{m(k+1)} \pmod n$, $B_{2mk-1}\equiv(-1)^{m(k+1)} \pmod n$, and by (22) $A''_{2mk}=B_{2mk}+MB_{2mk-1}$, and get on reducing,

$$B_{2mk}\equiv-B'_{2mk} \pmod n. \tag{65}$$

IX.

The partial quotient of order $2mk$ in continued fraction (2) is zero, modulo n, so that we may write the following congruences:

$$\left.\begin{aligned} &B'_{2mk}\equiv B'_{2mk-2},\\ &B'_{2mk-1}\equiv a_1B'_{2mk-2}+B'_{2mk-3}\equiv(-1)^{m(k+1)},\\ &B'_{2mk-2}\equiv a_2B'_{2mk-3}+B'_{2mk-4},\\ &\dots\dots\dots\dots\dots\dots\dots, \end{aligned}\right\} \pmod n$$

whence we obtain:

$$\left.\begin{aligned} &B'_{2mk}\equiv B'_{2mk}\equiv-B_{2mk}, \text{ by (65)},\\ &B'_{2mk-1}\equiv(-1)^{m(k+1)},\\ &B'_{2mk-2}\equiv-B_{2mk},\\ &B'_{2mk-3}\equiv-a_1(-B_{2mk})+(-1)^{m(k+1)},\\ &B'_{2mk-4}\equiv(a_1a_2+1)(-B_{2mk})-a_2(-1)^{m(k+1)},\\ &\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \end{aligned}\right\} \pmod n$$

From these congruences we infer the following which is easily established by complete induction

$$B'_{2mk-r}\equiv(-1)^r(-A_{r-2}B_{2mk}-(-1)^{m(k+1)}B_{r-2}) \pmod n. \tag{66}$$

Put $r=2mk-1$ in this last congruence and get

$$B'_1 \equiv A_{2mk-3}B_{2mk}+(-1)^{m(k+1)}B_{2mk-3} \pmod{n}. \tag{67}$$

But $B'_1=1$ and $A_{2mk-1}-a_{k-1}A_{2mk-2}=A_{2mk-3}$, $B_{2mk-1}-a_{k-1}B_{2mk-2}=B_{2mk-3}$, so that $A_{2mk-3}\equiv a_{k-1}(-1)^{m(k+1)} \pmod{n}$, and (67) reduces easily to

$$B_{2mk}\equiv -B_{2mk-2} \pmod{n}. \tag{68}$$

But since the partial quotient of (1) of rank $2mk$ is $-2M$, and $B_{2mk-1}\equiv(-1)^{mk+1}$, we have $B_{2mk}\equiv -2M(-1)^{mk+1}+B_{2mk-2} \pmod{n}$, so that, by (68),

$$B_{2mk}\equiv -M(-1)^{mk+1} \pmod{n}. \tag{69}$$

Returning with this value to (66) we get

$$B'_{2mk-r}\equiv(-1)^{m(k+1)+r+1}(B_{r-2}-MA_{r-2}) \pmod{n}. \tag{70}$$

X.

Reading backward from B_{2mk}, as in the derivation of (66) we arrive at the formula

$$B_{2mk-r}\equiv(-1)^{m(k+1)+r+1}(B'_{r-2}-MA'_{r-2}) \pmod{n}. \tag{71}$$

In this formula put $r=tk+1$; then

$$B_{2mk-tk-1}\equiv(-1)^{m(k+1)+tk}(B'_{tk-1}-MA'_{tk-1}) \pmod{n}, \tag{72}$$

and this with (63) gives $B_{2mk-tk-1}\equiv(-1)^{m(k+1)+tk-1}A''_{tk-1} \pmod{n}$. But, by (20) this gives

$$B_{2mk-tk-1}\equiv(-1)^{(m+t)(k+1)}B_{tk-1} \pmod{n}, \tag{73}$$

which is the same relation between the B's as (50) between the A's.

XI.

From (72) and (73) we get easily

$$B_{tk-1}\equiv(-1)^{t}(B'_{tk-1}-MA'_{tk-1}) \pmod{n}, \tag{74}$$

while (71) gives the corresponding formula:

$$B'_{tk-1}\equiv(-1)^{t}(B_{tk-1}-MA_{tk-1}) \pmod{n}. \tag{75}$$

XII.

Let m' now be the odd value of μ less than $2n$ which satisfies the congruence, $\mu b+2M\equiv 0 \pmod{n}$, and suppose first $2m$ is greater than this odd value so that $2m-m'=n$. We have

$$A_{2mk-1}\equiv 0\equiv A_{nk}A_{m'k-1}+A_{nk-1}B_{m'k-1} \pmod{n}, \tag{76}$$

$$A_{2mk} \equiv(-1)^{m(k+1)}\equiv A_{nk}A_{m'k}+A_{nk-1}B_{m'k} \pmod{n}. \tag{77}$$

Eliminate A_{nk-1} from these two equations and get

$$B_{m'k-1}\equiv A_{nk}(-1)^{k(m-m')+m} \pmod{n}. \tag{78}$$

But by (50) and (73) we know that

$$A_{nk-1}=(-1)^{(k+1)(m'+m)}A_{m'k-1} \pmod{n}, \tag{79}$$

$$B_{nk-1}\equiv(-1)^{(k+1)(m'+m)}B_{m'k-1} \pmod{n}, \tag{80}$$

so that

$$A_{nk}\equiv-B_{nk-1} \pmod{n}. \tag{81}$$

But $A_{2nk-1}=A_{nk}A_{nk-1}+B_{nk-1}A_{nk-1}=A_{nk-1}(A_{nk}+B_{nk-1})$, so that by (81) we have

$$A_{2nk-1}\equiv 0 \pmod{n}. \tag{82}$$

Also $B_{2nk}=A_{nk}B_{nk}+B_{nk}B_{nk-1}$; and this by (81) gives

$$B_{2nk}\equiv 0 \pmod{n}. \tag{83}$$

Again, $A_{2nk}=A_{nk}A_{nk}+A_{nk-1}B_{nk}\equiv -A_{nk}B_{nk-1}+A_{nk-1}B_{nk}$, by (81), and by (82) and (83) this gives

$$A_{2nk}\equiv(-1)^{nk-1} \pmod{n}. \tag{84}$$

And, finally, from the equation $A_{2nk}B_{2nk-1}-A_{2nk-1}B_{2nk}=1$, we derive

$$B_{2nk-1}\equiv(-1)^{nk-1} \pmod{n}. \tag{85}$$

XIII.

Suppose next that m' is greater than $2m$ so that $m'-2m=n$. We have then

$$A_{m'k}\equiv A_{nk}A_{2mk}+A_{nk-1}B_{2mk}, \quad A_{m'k-1}=A_{nk}A_{2mk-1}+A_{nk-1}B_{2mk-1}.$$

Put in these the values of A_{2mk}, etc., already derived, and these equations become

$$(-1)^{mk+1}A_{m'k} \equiv A_{nk}-MA_{nk-1} \pmod{n}. \tag{86}$$

$$(-1)^{mk+1}A_{m'k-1}\equiv A_{nk-1} \pmod{n}. \tag{87}$$

$$(-1)^{mk+1}B_{m'k} \equiv B_{nk}-MB_{nk-1} \pmod{n}. \tag{88}$$

$$(-1)^{mk+1}B_{m'k-1}\equiv B_{nk-1} \pmod{n}. \tag{89}$$

From these we get

$$A_{nk}+B_{nk-1}\equiv(-1)^{mk+1}(A_{mk}+MA_{mk-1}+B_{mk-1}). \tag{90}$$

But, recalling the definition of m' we observe that the fractions (1) and (2), as far as the $km'-1$-st term, are inverse modulo n, so that

$$A_{m'k-1}\equiv A'_{m'k-1} \pmod{n}. \tag{91}$$

$$A_{m'k-2}\equiv B'_{m'k-1} \pmod{n}. \tag{92}$$

$$B_{m'k-1}\equiv A'_{m'k-2} \pmod{n}. \tag{93}$$

$$B_{m'k-2}\equiv B'_{m'k-2} \pmod{n}. \tag{94}$$

Also by (5), since m' is odd, $-A_{m'k}=A_{m'k}+MA_{m'k-1}$.

Further, since the $m'k$-th partial quotient is zero in (2),

$$A_{mk}\equiv A_{mk-2} \pmod{n}, \tag{95}$$

whence, from (92) and (95)

$$A_{mk}+MA_{mk-1}\equiv -B_{mk-1} \pmod{n}, \tag{96}$$

which in (90) gives $A_{nk}+B_{nk-1}\equiv 0 \pmod{n}$, and this is the same formula as (81) for the case where $2m$ is greater than m'. Formulae (82), (83), (84) and (85) therefore hold whether $2m$ is greater or less than m'.

XIV.

From the equations

$$A_{4nk} = A_{2nk}+A_{2nk-1}B_{2nk}, \tag{97}$$

$$A_{4nk-1}=A_{2nk}A_{2nk-1}+A_{2nk-1}B_{2nk}, \tag{98}$$

$$B_{4nk} = B_{2nk}A_{2nk}+B_{2nk-1}B_{2nk}, \tag{99}$$

$$B_{4nk-1}=B_{2nk}A_{2nk-1}+B_{2nk-1}B_{2nk-1}, \tag{100}$$

we now derive

$$A_{4nk} \equiv B_{4nk-1}\equiv 1 \pmod{n}, \tag{101}$$

$$A_{4nk-1}\equiv B_{4nk} \equiv 0 \pmod{n}. \tag{102}$$

From these results it appears that taken modulo n the series of convergents repeat themselves with a period of $4nk$, but if k is odd as well as n, the period is $2nk$.

XV.

We now extend the above results to the fraction

$$\overline{(a_1, a_2, a_3, \ldots, a_{k-1}, \mu b+c)}, \quad (\mu=1, 2, 3, \ldots), \tag{103}$$

where, as before, n is prime to $2b$, and to A_{k-1} and B_{k-1}. It is clear that there will be a partial quotient $\mu b+c\equiv 0 \pmod{n}$, after which the fraction is of the type (1). Call P_r/Q_r the r-th convergent of (103) and as before A_r/B_r the r-th convergent of (1). Then we have, μ being determined by the congruence $\mu b+c\equiv 0 \pmod{n}$,

$$P_{\mu+2nk-1}\equiv P_{\mu k}A_{2nk-1}+P_{\mu k-1}B_{2nk-1} \pmod{n}. \tag{104}$$

$$P_{\mu+2nk} \equiv P_{\mu k}A_{2nk}+P_{\mu k-1}B_{2nk} \pmod{n}. \tag{105}$$

$$Q_{\mu+2nk-1}\equiv Q_{\mu k}A_{2nk-1}Q_{\mu k-1}B_{2nk-1} \pmod{n}. \tag{106}$$

$$Q_{\mu+2nk} \equiv Q_{\mu k}A_{2nk}+Q_{\mu k-1}B_{2nk} \pmod{n}. \tag{107}$$

But we have also,

$$P_{\mu+2nk-1}\equiv P_{2nk}P_{\mu k-1}+P_{2nk-1}Q_{\mu k-1} \pmod{n}. \tag{108}$$

$$P_{\mu+2nk} \equiv P_{2nk}P_{\mu k}+P_{2nk-1}Q_{\mu k} \pmod{n}. \tag{109}$$

$$Q_{\mu+2nk-1}\equiv Q_{2nk}P_{\mu k-1}+Q_{2nk-1}Q_{\mu k-1} \pmod{n}. \tag{110}$$

$$Q_{\mu+2nk} \equiv Q_{2nk}P_{\mu k}+Q_{2nk-1}Q_{\mu k} \pmod{n}. \tag{111}$$

Using (82), (83), (84) and (85), we derive from these,

$$P_{\mu k-1}[(-1)^{nk-1}-P_{2nk}]\equiv P_{2nk-1}Q_{\mu k-1} \pmod{n}. \tag{112}$$

$$P_{\mu k}[(-1)^{nk-1}-P_{2nk}] \equiv P_{2nk-1}Q_{\mu k} \pmod{n}. \tag{113}$$

$$Q_{\mu k-1}[(-1)^{nk-1}-Q_{2nk-1}]\equiv Q_{2nk}P_{\mu k-1} \pmod{n}. \tag{114}$$

$$Q_{\mu k}[(-1)^{nk-1}Q_{2nk-1}]\equiv Q_{2nk}P_{\mu k} \pmod{n}. \tag{115}$$

Eliminate now P_{2nk-1} from (112) and (113) and obtain

$$(P_{\mu k-1}Q_{\mu k}-P_{\mu k}P_{\mu k-1})[(-1)^{nk-1}-P_{2nk}]\equiv 0 \pmod{n}.$$

Then, since the first factor on the left is ± 1, we have

$$P_{2nk}\equiv(-1)^{nk-1} \pmod{n}. \tag{116}$$

Similarly, from (114) and (115) we get,

$$Q_{2nk-1}\equiv(-1)^{nk-1} \pmod{n}. \tag{117}$$

Also, by (112) and (113), $P_{2nk-1}Q_{\mu k-1}\equiv 0 \pmod{n}$, $P_{2nk-1}Q_{\mu k}\equiv 0 \pmod{n}$, and since $Q_{\mu k}$ and $Q_{\mu k-1}$ can not both be congruent to zero on account of the equation $P_{\mu k}Q_{\mu k-1}-P_{\mu k-1}Q_{\mu k}=(-1)^{\mu k}$, we must have

$$P_{2nk-1}\equiv 0 \pmod{n}, \tag{118}$$

and, similarly,

$$Q_{2nk}\equiv 0 \pmod{n}. \tag{119}$$

XVI.

The above results may be extended to fractions which have a set of "irregular" or non-periodic partial quotients followed by partial quotients of the sort considered in fraction (103). Such a fraction would be of the form

$$(q_1, q_2, \ldots, q_r, \overline{a_1, a_2, \ldots, a_{k-1}, \mu b+c}), \qquad (\mu=1, 2, 3, \ldots).$$

For this fraction there will be r non-periodic convergents, after which the periodicity begins, the length of the period being the same as for the fraction (103). The two successive convergents which close each period will, however, not be congruent respectively to $\pm 1, 0$ and $0, \pm 1$, but to the $(r-1)$-st and the r-th convergents respectively of the fraction $(q_1, q_2, \ldots, q_r)$.

XVII.

We consider now the special cases which have been ruled out in the statement of our theorems, and take up first the case where b is congruent to zero, modulo n. This will include also the case where b is actually zero, in which case the fraction is an ordinary purely periodic continued fraction. We may write it in the form $\overline{(a_1, a_2, a_3, \ldots, a_k)}$. The convergents of order k and $k-1$ are connected with those of order $2k$ and $2k-1$ by the following equations:

$$\begin{aligned} A_{2k} &= A_k^2+A_{k-1}B_k, & B_{2k} &= B_kA_k+B_{k-1}B_k, \\ A_{2k-1} &= A_kA_{k-1}+A_{k-1}B_{k-1}, & B_{2k-1} &= B_kA_{k-1}+B_{k-1}^2. \end{aligned}$$

Now, these are seen to result from the same process as that by which two linear homogeneous substitutions are compounded, so that if we call T_k the substitution,

$$T_k=\begin{pmatrix} A_k, & B_k \\ A_{k-1}, & B_{k-1} \end{pmatrix},$$

we have $T_{2k}=T_k^2$, and in general, $T_{rk}=T_k^r$. Now the theory of the periodicity of such a substitution modulo a prime is well known. Gierster has shown (*Math. Ann.*, Vol. XVIII, p. 322) that the period with respect to a prime modulus p of $\begin{pmatrix}\alpha, & \beta\\ \gamma, & \delta\end{pmatrix}$, where $\alpha\delta-\beta\gamma\equiv 1 \pmod p$, depends upon the quadratic character of $\Delta=(\alpha+\delta)^2/4-1$ with respect to p. In fact, if we denote the quadratic character by ε, where ε is $+1$ if Δ is a quadratic residue of p; -1 if Δ is a quadratic non-residue of p, and 0 if Δ is divisible by p, then the period of $\begin{pmatrix}\alpha, & \beta\\ \gamma, & \delta\end{pmatrix}$ is a divisor of $p-\varepsilon$. More narrowly, the period is a divisor of $(p-\varepsilon)/2$ if p is not a divisor of Δ, and the period is p if p is a divisor of Δ. In Gierster's discussion the period would be considered closed when some power of the substitution should be of the form $\begin{pmatrix}x, & 0\\ 0, & x\end{pmatrix}$, since in that case the x could be divided out of the fractional substitution. Manifestly in the application of his results to our problem no such division is possible. Using Gierster's results we may state the following theorem:

The purely periodic continued fraction $\overline{(a_1, a_2, \ldots, a_k)}$ *taken with respect to a prime, modulus* p*, recurs with a period* $\delta\pi k$ *where* δ *is some divisor of* $p-\varepsilon$*, where* $\varepsilon=\pm 1$ *or zero according as* $\Delta=(A_k+B_{k-1})^2/4-(-1)^k$ *is a quadratic residue, a quadratic non-residue, or a divisor of* p*, while* π *is the exponent to which* $A_{k\delta}$ *belongs, modulo* p.

XVIII.

In congruence (37) we have restricted n to be prime to $2b$. For odd values of n all of our results are still valid if n and $2b$ are not relatively prime, provided that the congruence (37) has roots; that is, provided the greatest common divisor of n and b also divides $2M'$ where M' is an integer congruent modulo m to the fraction M. (The existence of such an integer M' follows from the fact that we are still supposing n to be prime to the denominator A_{k-1} of M). If the congruence (37) does not have roots the discussion given above does not apply, and the intrinsic importance of congruence (37) in the theory appears when we observe that if it has no roots the general results which we have obtained no longer hold, as the numerical example $\overline{(3, 2, 1, 4, 7m)}$ (mod 21) shows. If the congruence (37) has no roots we compute the period of the partial quotients and apply the less definite method of the preceding paragraph.

For even values of n the further difficulty arises in that while (37) may have roots, the values of the unknown will all be even, or all odd, so that the results of XII and XIII do not apply. By an analysis similar to that used in XII we may show, however, that formulae (82), (83), (84) and (85) hold for this case also. They do not hold if (37) has no roots.

XIX.

In the case where A_{k-1} is congruent to zero, modulo n, the congruence (37) becomes meaningless by the definition of M. The sequence of A's is, however, easily determined. For from the equation, $A_{k+r}=A_kA_r+A_{k-1}B_r$, we get, $A_{k+r}\equiv A_kA_r \pmod n$, so that the list of A's from A_{k+1} to A_{2k} is obtained by multiplying the list from A_1 to A_k by A_k, and so on. In particular, the list of values of A_{rk-1}, $(r=1, 2, 3, \ldots)$ are all zero, while $A_{rk}\equiv A_k^r \pmod n$, from which, by Fermat's Theorem, $A_{2nk}\equiv A_k^{2n}\equiv A_k^2 \pmod n$. From this congruence it is clear that unless A_k belongs to the exponent 2, modulo n, the period of the convergents can not close with the one of order $2nk$. If g is the exponent to which A_k belongs, modulo n, then the period of the convergents must be a multiple of gnk.

XX.

In conclusion we apply the foregoing results to establish the theorems mentioned in the introduction which relate to the continued fraction for $e=(2, 1, 2, 1, 1, 4, 1, 1, 6, \ldots)$. The convergents to this fraction are connected with those of the fraction $e-1=(1, 1, 2, 1, 1, 4, \ldots)$ by the simple equations, easily established, $A_m-B_m=A'_m$, $B_m=B'_m$, where the convergent A/B refers to the first fraction and A'/B', to the second. Now for the second fraction the value of M is found to be unity, so that congruence (37) becomes $2\mu+2\equiv 0 \pmod n$. Suppose first that n is odd. The congruence has the root $\mu\equiv -1$, so the two values of μ less than $2n$ which satisfy this congruence are $n-1$ and $2n-1$, of which the first is even. Put this for $2m$ in (38), (47), (49) and (69), and remember that $k=3$ in this fraction and get

$$A'_{3n-4}\equiv 0 \pmod n,\qquad B'_{3n-4}\equiv 1 \pmod n,$$
$$A'_{3n-3}\equiv 1 \pmod n,\qquad B'_{3n-3}\equiv -1 \pmod n.$$

From these, knowing that $a_{3n-2}=1$ and $a_{3n-4}=1$ and $a_{3n}\equiv 0 \pmod n$, we readily compute

$$A'_{3n}\equiv 1 \pmod n,\qquad B'_{3n}\equiv 0 \pmod n,$$
$$A'_{3n-1}\equiv 2 \pmod n,\qquad B'_{3n-1}\equiv -1 \pmod n,$$

whence

$$A_{3n}\equiv 1 \pmod n,\qquad B_{3n}\equiv 0 \pmod n,$$
$$A_{3n-1}\equiv 1 \pmod n,\qquad B_{3n-1}\equiv -1 \pmod n.$$

From these follow the rest of the results mentioned. The results for even values of n are easily obtained.